EC400: SOFP

Class #3

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Problem 1

Problem 1 [Harder]

For $a > 0$, consider the problem:

$$\max_{x,y} ax + y$$
$$\text{s.t. } x^2 + (x - y)^2 \leq 1$$
$$x \geq a,\, y \geq 0.$$

Using the Kuhn-Tucker approach, write down the necessary first order conditions that must be satisfied by the solution of the constrained optimization problem. Are solutions to these conditions also maximizers of the Lagrangian? Solve the constrained optimization problem in terms of a. Find the slope of value function with respect to a, possibly relying on the Envelope Theorem.

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  - It collapses to a singleton when $a = 1$, and
  - it is an empty set when $a > 1$.

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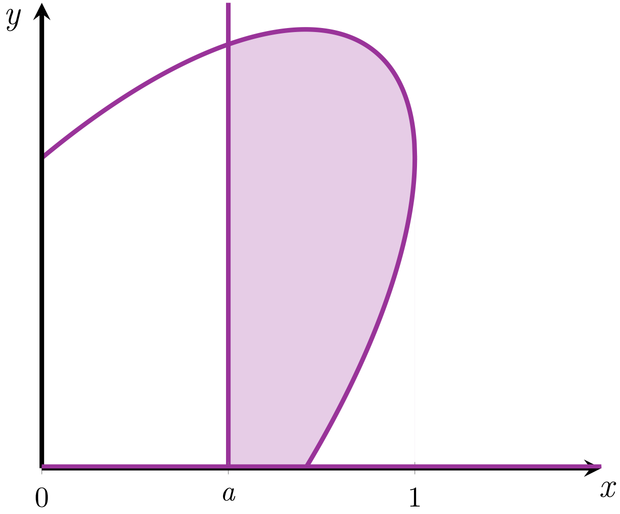

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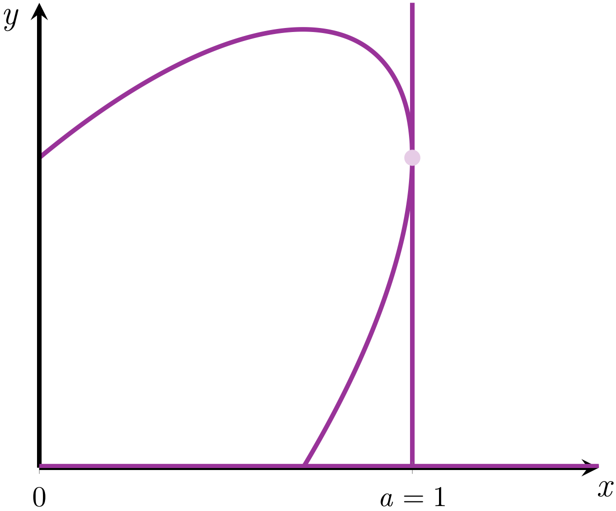

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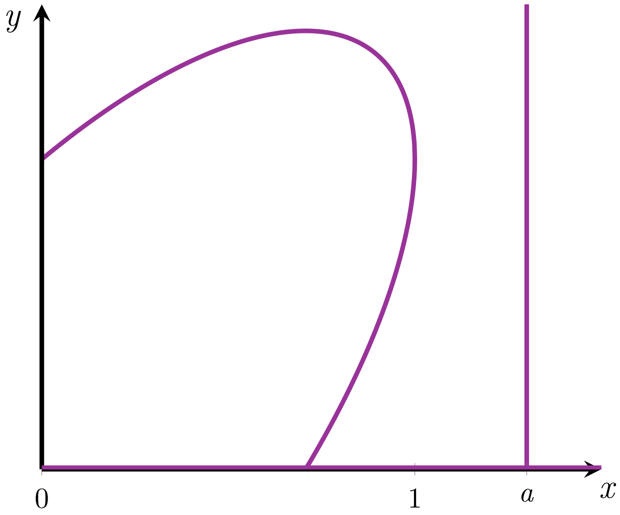

Problem 1 [Harder]

- Lagrangian:

$$\mathcal{L}(x, y, \boldsymbol{\lambda} \mid a) = ax + y - \lambda_1(x^2 + (x-y)^2 - 1) - \lambda_2(a - x) + \lambda_3 y$$

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- Necessary FOC:

$$[x] \qquad a - \lambda_1^*(4x^* - 2y^*) + \lambda_2^* = 0$$

$$[y] \qquad 1 - 2\lambda_1^*(y^* - x^*) + \lambda_3^* = 0$$

$$[\lambda_1] \qquad \lambda_1^* \geq 0, x^{*2} + (x^* - y^*)^2 \leq 1, \lambda_1^* \left[x^{*2} + (x^* - y^*)^2 - 1\right] = 0$$

$$[\lambda_2] \qquad \lambda_2^* \geq 0, x^* \geq a, \lambda_2^*(x^* - a) = 0$$

$$[\lambda_3] \qquad \lambda_3^* \geq 0, y^* \geq 0, \lambda_3^* y^* = 0$$

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- To solve the system, let's consider all the possible cases in terms of which constraints are binding.

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- Suppose $\lambda_3^* > 0$. Then, $y^* = 0$. From $[y]$: $\lambda_3^* = -(1 + 2\lambda_1^* x^*) < 0$ since $\lambda_1^* > 0$ and $x^* \geq a > 0$. Hence, $\boxed{\lambda_3^* = 0}$.

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  - For this solution to be well-defined: $1 - a^2 > 0 \iff a < 1$ (remember $a > 0$) which holds under the Slater condition.

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  - From $[y]$: $\boxed{\lambda_1^* = \frac{1}{2\sqrt{1-a^2}}} > 0$ ✓.

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  - For this solution to be well-defined: $1 - a^2 > 0 \iff a < 1$ (remember $a > 0$) which holds under the Slater condition.
  - From $[y]$: $\boxed{\lambda_1^* = \frac{1}{2\sqrt{1-a^2}}} > 0$ ✓.
  - From $[x]$: $\boxed{\lambda_2^* = \frac{a}{\sqrt{1-a^2}} - 1 - a} \geq 0 \iff \boxed{a \geq 0.8832}$. So, this is a **solution for sufficiently high a.**

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  - From $[\lambda_1]$: $\boxed{x^* = \frac{1+a}{\sqrt{(1+a)^2+1}}} \implies \boxed{y^* = \frac{2+a}{\sqrt{(1+a)^2+1}}}$.

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  - Then, $\boxed{\lambda_1^* = \frac{\sqrt{(1+a^2)+1}}{2}} > 0$ ✓.

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  - From $[\lambda_1]$: $\boxed{x^* = \frac{1+a}{\sqrt{(1+a)^2+1}}} \implies \boxed{y^* = \frac{2+a}{\sqrt{(1+a)^2+1}}}$.
  - Then, $\boxed{\lambda_1^* = \frac{\sqrt{(1+a^2)+1}}{2}} > 0$ ✓.
  - Finally, check $x^*(a) \geq a \iff \boxed{a \leq 0.8832}$. So, this is a **solution for sufficiently small a.**

Problem 1 [Harder]

- Envelope theorem: If $(x^*(a), y^*(a), \boldsymbol{\lambda}^*(a))$ are differentiable and $(x^*(a), y^*(a))$ satisfies the constraint qualification, then

$$\frac{\mathrm{d}v(a)}{\mathrm{d}a} = \frac{\partial \mathcal{L}(x^*(a), y^*(a), \boldsymbol{\lambda}^*(a) \mid a)}{\partial a}$$

$$= x^*(a) - \lambda_2^*(a)$$

$$= \begin{cases} \frac{1+a}{\sqrt{(1+a)^2+1}} & \text{if } a \leq 0.8832 \\ 2a + 1 - \frac{a}{\sqrt{1-a^2}} & \text{if } a \geq 0.8832 \end{cases}.$$

Problem 5

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For each of the following correspondences $h : \mathbb{R} \rightrightarrows \mathbb{R}$ show whether they are convex- valued, upper-hemicontinuous or lower-hemicontinuous:

(a) $h(x) = [5x, 10x)$ for $x \in [0, 1]$;

(b) $h(x) = \{5x, 10x\}$ for $x \in [0, 1]$;

(c) $$h(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0, 1) \\ [0, 1] & \text{for } x = 1 \end{cases}.$$

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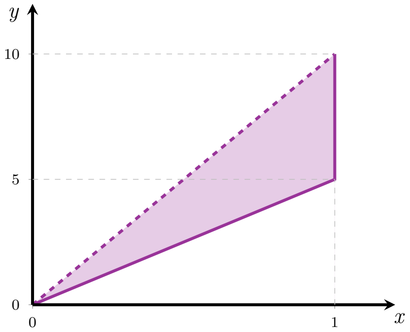
Solution: Part (a)

$$h(x) = [5x, 10x) \text{ for } x \in [0, 1]$$



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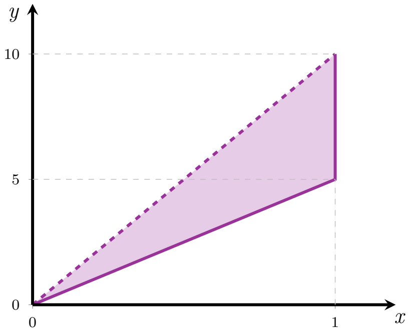

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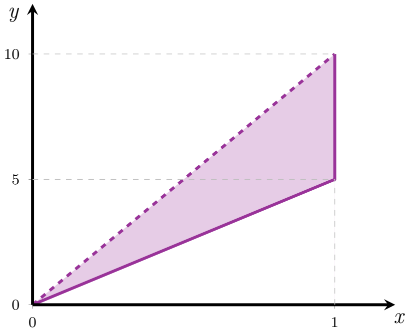

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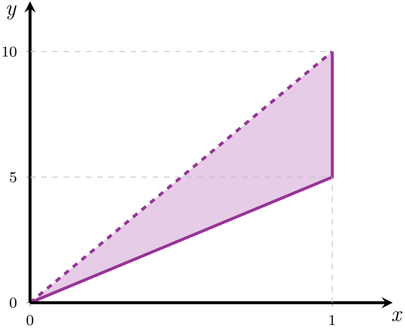

- Open graph $\implies$ **not UHC**.
- Set $[5x, 10x)$ convex $\forall\, x \in [0,1]$ $\implies$ **convex-valued**.
- $\forall\,(x,y) \in \mathrm{Gr}\,(h)$, $\forall\, \{x_k\}_{k=1}^{\infty} : x_k \to x$, $\exists\, \{y_k\}_{k=1}^{\infty} : y_k \in [5x, 10x)\ \forall\, k \in \mathbb{N}$ and $y_k \to y$. Thus, **LHC**.

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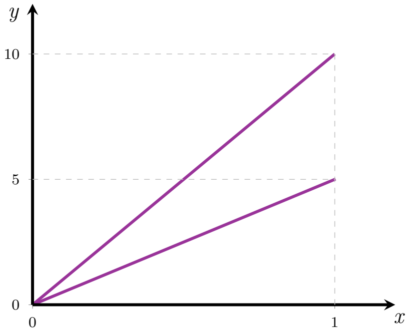

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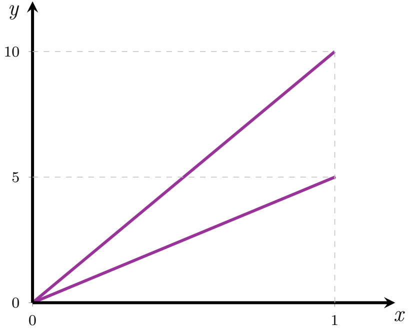

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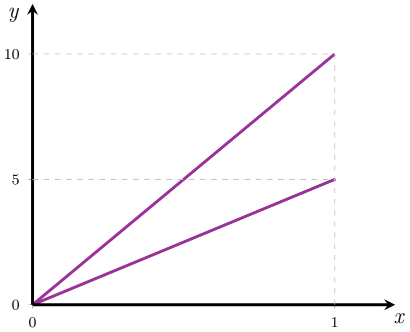

- Graph closed and union of continuous functions $\implies$ both **UHC** and **LHC**.
- $\nexists x \in [0,1]$: set $\{5x, 10x\}$ convex $\implies$ **not convex-valued**.

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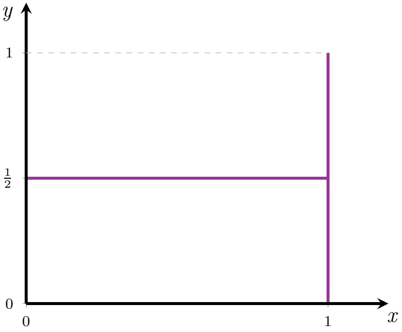
Solution: Part (c)

$$h(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0,1) \\ [0,1] & \text{for } x = 1 \end{cases}$$



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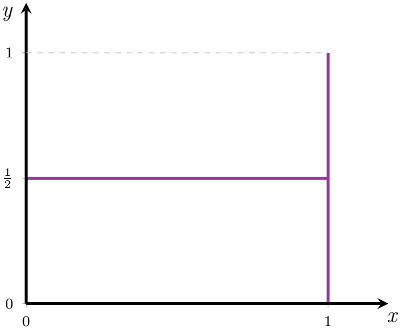

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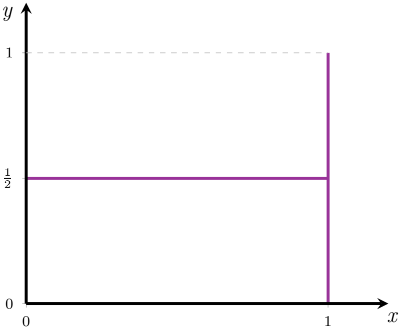

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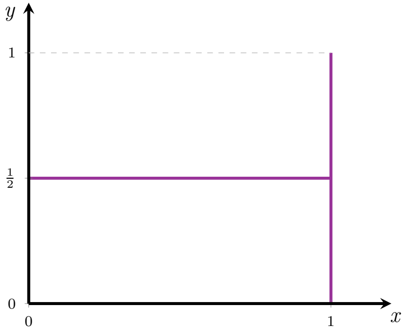

- Graph closed $\implies$ **UHC**.
- Sets $\{\frac{1}{2}\}$ and $[0,1]$ convex $\implies$ **convex-valued**.
- $(1,1) \in \text{Gr}(h)$, but $\forall \{x_k\}_{k=1}^{\infty} : x_k \to 1$, $y_k \to \frac{1}{2}$. Thus, **not LHC**.

Problem 7

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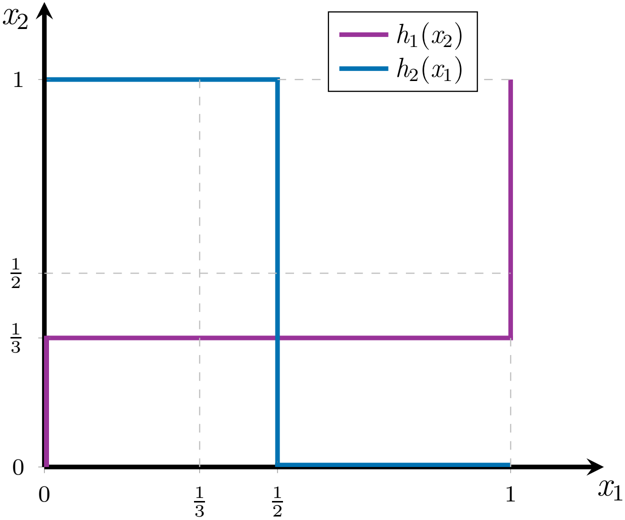
Consider the correspondence $h : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, where $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), h_2(\mathbf{x}))$,

$$h_1(\mathbf{x}) = \begin{cases} \{1\} & \text{if } x_2 > \frac{1}{3} \\ [0,1] & \text{if } x_2 = \frac{1}{3} \\ \{0\} & \text{if } x_2 < \frac{1}{3} \end{cases} \quad \text{and} \quad h_2(\mathbf{x}) = \begin{cases} \{1\} & \text{if } x_1 < \frac{1}{2} \\ [0,1] & \text{if } x_1 = \frac{1}{2} \\ \{0\} & \text{if } x_1 > \frac{1}{2} \end{cases}.$$

Show how you can expolit Kakutani's Fixed Point Theorem despite the domain being open, and find all the fixed points.

Problem 7
Solution



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- Both correspondences have closed graphs $\implies$ **UHC**. Both are obviously **non-empty** and **convex-valued**.
- The image of $\mathbf{h}$ belongs to $[0,1]^2$. We can restrict the domain to $[0,1]^2$ and Kakutani's FP theorem applies: $\exists$ FP in $[0,1]^2$.
- Moreover, $\nexists$ FP in $\mathbb{R}^2 \setminus [0,1]^2$ since the image is in $[0,1]^2$.

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  - $x_2 = \frac{1}{3} \implies h_1(x_2) = [0, 1] \implies x_2 \in h_1(x_2)$.

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  - Thus, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is a FP.
  - It is the only FP.