EC400: SOFP

Class #2

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Problem 1

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Find critical points, local maxima and local minima for each of the following functions:

(a) $x^4 + x^2 - 6xy + 3y^2$

(b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

(c) $xy^2 + x^3y - xy$

Which of the critical points are also global maxima or global minima?

Problem 1

Solution: Part (a)

- Find critical points:

$$Df(x^*, y^*)' = \begin{pmatrix} 4x^{*3} + 2x^* - 6y^* \\ -6x^* + 6y^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies (x^*, y^*) \in \{(0,0), (1,1), (-1,-1)\}.$$

details

Problem 1
Solution: Part (a)

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- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}.$$

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Solution: Part (a)

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- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

- The LPMs are:

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Solution: Part (a)

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- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}.$$

- The LPMs are:
  - $\det(L_1) = \det\left(12x^2+2\right) = 12x^2+2.$

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Solution: Part (a)

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- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

- The LPMs are:
  - $\det(L_1) = \det\left(12x^2 + 2\right) = 12x^2 + 2.$
  - $\det(L_2) = \det\left(D^2 f(x, y)\right) = 6(12x^2 + 2) - 36 = 72x^2 - 24.$

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Solution: Part (a)

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- Check definiteness of the Hessian at each critical point:

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- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 2 > 0$ and $|L_2| = -24 < 0 \implies$ indefinite $\implies$ **saddle point.**

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Solution: Part (a)

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- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 2 > 0$ and $|L_2| = -24 < 0 \implies$ indefinite $\implies$ **saddle point.**
  - At $(1,1)$: $|L_1| = 14 > 0$ and $|L_2| = 48 > 0 \implies$ p.d. $\implies$ **strict local min.**

Problem 1
Solution: Part (a)

- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 2 > 0$ and $|L_2| = -24 < 0 \implies$ indefinite $\implies$ **saddle point.**
  - At $(1,1)$: $|L_1| = 14 > 0$ and $|L_2| = 48 > 0 \implies$ p.d. $\implies$ **strict local min.**
  - At $(-1,-1)$: $|L_1| = 14 > 0$ and $|L_2| = 48 > 0 \implies$ p.d. $\implies$ **strict local min.**

Problem 1
Solution: Part (a)

- Look for global max:

Problem 1
Solution: Part (a)

- Look for global max:
  - The function is unbounded above $\Longrightarrow$ **no global max**:

$$f(x, x) = x^4 - 2x \underset{x \to \infty}{\longrightarrow} \infty.$$

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Solution: Part (a)

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- Look for global max:
  - The function is unbounded above $\implies$ **no global max**:

$$f(x,x) = x^4 - 2x \underset{x\to\infty}{\longrightarrow} \infty.$$

- Look for global min:

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Solution: Part (a)

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- Look for global max:
  - The function is unbounded above $\implies$ **no global max**:

$$f(x,x) = x^4 - 2x \underset{x\to\infty}{\longrightarrow} \infty.$$

- Look for global min:
  - $(1,1)$ and $(-1,-1)$ are **global minima**:

$$f(x,y) = \underbrace{3(x-y)^2}_{\geq 0} + \underbrace{x^2(x^2-2)}_{\text{min at } x^2=1}.$$

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Solution: Part (b)

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- Find critical points:

$$Df(x^*, y^*)' = \begin{pmatrix} 2x^* - 6y^* + 10 \\ -6x^* + 4y^* + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies (x^*, y^*) \in \left\{ \left(\frac{13}{7}, \frac{16}{7} \right) \right\}.$$

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Solution: Part (b)

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- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}.$$

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Solution: Part (b)

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- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}.$$

- **Not a local extremum** since the Hessian is indefinite: The LPMs are

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Solution: Part (b)

- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}.$$

- **Not a local extremum** since the Hessian is indefinite: The LPMs are
  - $\det(L_1) = \det\left(2\right) = 2 > 0 \ \forall (x, y) \in \mathbb{R}^2$.

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Solution: Part (b)

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- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}.$$

- **Not a local extremum** since the Hessian is indefinite: The LPMs are
  - $\det(L_1) = \det(2) = 2 > 0 \; \forall (x, y) \in \mathbb{R}^2$.
  - $\det(L_2) = \det\left(D^2 f(x, y)\right) = 8 - 36 = -28 < 0 \; \forall (x, y) \in \mathbb{R}^2$.

Problem 1

Solution: Part (b)

- **No global max** since the function is unbounded from above:

$$\lim_{x \to \infty} f(x, 0) = \lim_{x \to \infty} x^2 + 10x - 5 = +\infty.$$

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Solution: Part (b)

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- **No global max** since the function is unbounded from above:

$$\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} x^2 + 10x - 5 = +\infty.$$

- **No global min** since the function is unbounded from below:

$$\lim_{x\to\infty} f(x,x) = \lim_{x\to\infty} -3x^2 + 12x - 5 = -\infty.$$

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Solution: Part (c)

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- Find critical points:

$$Df\left(x^*, y^*\right)' = \begin{pmatrix} {y^*}^2 + 3{x^*}^2 y^* - y^* \\ 2x^* y^* + 6{x^*}^3 - x^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \left(x^*, y^*\right) \in \left\{ (0,0), (0,1), (-1,0), (1,0), \left(-\frac{1}{\sqrt{5}}, \frac{2}{5}\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right) \right\}.$$

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- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

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- Obtain the Hessian:

$$D^2 f(x, y) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

- The LPMs are:

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Solution: Part (c)

- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

- The LPMs are:
  - $\det(L_1) = \det\big(6xy\big) = 6xy$.

Problem 1

Solution: Part (c)

- Obtain the Hessian:

$$D^2 f(x,y) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

- The LPMs are:
  - $\det(L_1) = \det\big(6xy\big) = 6xy.$
  - $\det(L_2) = \det\big(D^2 f(x,y)\big) = 12x^2y - (2y + 3x^2 - 1)^2 = -9x^4 + 6x^2 - 4y^2 + 4y - 1.$

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- Check definiteness of the Hessian at each critical point:

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- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**

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- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**
  - At $(0,1)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**

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Solution: Part (c)

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- Check definiteness of the Hessian at each critical point:
  - At $(0,0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**
  - At $(0,1)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**
  - At $(1,0)$: $|L_1| = 0$ and $|L_2| = -4 < 0 \implies$ indefinite $\implies$ **saddle point.**

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- At $(-1, 0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**

Problem 1

Solution: Part (c)

- At $(-1, 0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**
- At $\left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$: $|L_1| = 12/5^{3/2} > 0$ and $|L_2| = 4/5 > 0 \implies$ p.d. $\implies$ **strict local min.**

Problem 1

Solution: Part (c)

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- At $(-1, 0)$: $|L_1| = 0$ and $|L_2| = -1 < 0 \implies$ indefinite $\implies$ **saddle point.**
- At $\left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$: $|L_1| = 12/5^{3/2} > 0$ and $|L_2| = 4/5 > 0 \implies$ p.d. $\implies$ **strict local min.**
- At $\left(-\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$: $|L_1| = -12/5^{3/2} < 0$ and $|L_2| = 4/5 > 0 \implies$ n.d. $\implies$ **strict local max.**

Problem 1
Solution: Part (b)

- **No global max** since the function is unbounded from above:

$$\lim_{x \to \infty} f(x, x) = \lim_{x \to \infty} x^3 + x^4 - x^2 = +\infty.$$

Problem 1

Solution: Part (b)

- **No global max** since the function is unbounded from above:

$$\lim_{x\to\infty} f(x,x) = \lim_{x\to\infty} x^3 + x^4 - x^2 = +\infty.$$

- **No global min** since the function is unbounded from below:

$$\lim_{x\to\infty} f(x,-x) = \lim_{x\to\infty} x^3 - x^4 + x^2 = -\infty. \quad \square$$

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Problem 2

Let $S \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}$ be a twice-continuously differentiable function. Suppose that $Df(\mathbf{x}^*) = 0$. State the weakest sufficient conditions that the Hessian must satisfy at the critical point $\mathbf{x}^*$ for:

1. $\mathbf{x}^*$ to be a local max;
2. $\mathbf{x}^*$ to be a strict local min.

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1. For a local max:

$$\exists\, \varepsilon > 0 : D^2 f(\mathbf{x}) \leq 0 \;\forall\, x \in B(\mathbf{x}^* \mid \varepsilon)$$

where $B(\mathbf{x}^* \mid \varepsilon) \equiv \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y} - \mathbf{x}|| < \varepsilon\}$.

□

Problem 2
Solution

1. For a local max:

$$\exists \varepsilon > 0 : D^2 f(\mathbf{x}) \leq 0 \, \forall x \in B(\mathbf{x}^* \mid \varepsilon)$$

where $B(\mathbf{x}^* \mid \varepsilon) \equiv \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y} - \mathbf{x}|| < \varepsilon\}$.

2. For a strict local min:

$$D^2 f(\mathbf{x}^*) \geq 0 \text{ and } \exists \varepsilon > 0 : D^2 f(\mathbf{x}) > 0 \, \forall x \in \overline{B}(\mathbf{x}^* \mid \varepsilon)$$

where $\overline{B}(\mathbf{x}^* \mid \varepsilon) \equiv B(\mathbf{x}^* \mid \varepsilon) \setminus \{\mathbf{x}^*\}$.

□

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Check whether $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is concave or convex by using the Hessian.

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Obtain the Hessian:

- $$Df(x,y)' = \begin{pmatrix} 4x^3 + 2xy^2 - 3 \\ 2x^2y + 4y^3 - 8 \end{pmatrix}$$

Problem 3
Solution

Obtain the Hessian:

- $$Df(x,y)' = \begin{pmatrix} 4x^3 + 2xy^2 - 3 \\ 2x^2y + 4y^3 - 8 \end{pmatrix}$$

- $$D^2f(x,y) = \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix}$$

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Check the principal minors:

- $$\det \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix} = 24(x^4 + y^4) + 132x^2y^2 \geq 0,$$

Check the principal minors:

- $\det\begin{pmatrix} 12x^2+2y^2 & 4xy \\ 4xy & 2x^2+12y^2 \end{pmatrix} = 24(x^4+y^4)+132x^2y^2 \geq 0,$
- $\det\left(12x^2+2y^2\right) = 12x^2+2y^2 \geq 0$, and

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Check the principal minors:

- $\det \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix} = 24(x^4 + y^4) + 132x^2y^2 \geq 0$,
- $\det \left(12x^2 + 2y^2\right) = 12x^2 + 2y^2 \geq 0$, and
- $\det \left(2x^2 + 12y^2\right) = 2x^2 + 12y^2 \geq 0$.

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Check the principal minors:

- $\det \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix} = 24(x^4 + y^4) + 132x^2y^2 \geq 0,$
- $\det \left(12x^2 + 2y^2\right) = 12x^2 + 2y^2 \geq 0$, and
- $\det \left(2x^2 + 12y^2\right) = 2x^2 + 12y^2 \geq 0$.
- All principal minors $\geq 0 \implies$ Hessian is p.s.d. $\implies f(\cdot)$ is convex on $\mathbb{R}^2$.

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Problem 4 [Harder]

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Solve the following problem

$$\max_{x,y} \left[\min\{x, y\} - x^2 - y^2\right].$$

Problem 4 [Harder]
Solution

- Can't use FOC since **differentiability fails**. What can we do?

Problem 4 [Harder]
Solution

- Can't use FOC since **differentiability fails**. What can we do?
- It turns out we can easily show that $x^* = y^*$ at any solution.

Problem 4 [Harder]
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- Can't use FOC since **differentiability fails**. What can we do?
- It turns out we can easily show that $x^* = y^*$ at any solution.
- Let $f(x, y) = \min\{x, y\} - x^2 - y^2$ and WLOG suppose that $x^* > y^*$.

Problem 4 [Harder]
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- Let $z^* = \frac{x^*+y^*}{2}$ and note that

Problem 4 [Harder]
Solution

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- Let $z^* = \frac{x^*+y^*}{2}$ and note that
  - $y^* + y^* < x^* + y^* < x^* + x^*$

Problem 4 [Harder]
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- Let $z^* = \frac{x^*+y^*}{2}$ and note that
  - $y^* + y^* < x^* + y^* < x^* + x^* \iff y^* < z^* < x^*$.

Problem 4 [Harder]
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- Let $z^* = \frac{x^*+y^*}{2}$ and note that
  - $y^* + y^* < x^* + y^* < x^* + x^* \iff y^* < z^* < x^*$.
  - By concavity of $-x^2$ and Jensen's inequality,

$$-\frac{1}{2}x^{*2} - \frac{1}{2}y^{*2} < -\left(\frac{x^* + y^*}{2}\right)^2$$

Problem 4 [Harder]
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- Let $z^* = \frac{x^*+y^*}{2}$ and note that
  - $y^* + y^* < x^* + y^* < x^* + x^* \iff y^* < z^* < x^*$.
  - By concavity of $-x^2$ and Jensen's inequality,

$$-\frac{1}{2}x^{*2} - \frac{1}{2}y^{*2} < -\left(\frac{x^* + y^*}{2}\right)^2$$

$$\iff -x^{*2} - y^{*2} < -2z^{*2}.$$

Problem 4 [Harder]
Solution

- Therefore,

$$f(x^*, y^*) = \min\{x^*, y^*\} - x^{*2} - y^{*2}$$

Problem 4 [Harder]

Solution

- Therefore,

$$\begin{aligned} f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\ &= y^* - x^{*2} - y^{*2} \end{aligned}$$

Problem 4 [Harder]
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- Therefore,

$$\begin{aligned} f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\ &= y^* - x^{*2} - y^{*2} \\ &< z^* - x^{*2} - y^{*2} \end{aligned}$$

Problem 4 [Harder]
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- Therefore,

$$\begin{aligned} f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\ &= y^* - x^{*2} - y^{*2} \\ &< z^* - x^{*2} - y^{*2} \\ &< z^* - 2z^{*2} \end{aligned}$$

Problem 4 [Harder]
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- Therefore,

$$\begin{aligned} f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\ &= y^* - x^{*2} - y^{*2} \\ &< z^* - x^{*2} - y^{*2} \\ &< z^* - 2z^{*2} \\ &= \min\{z^*, z^*\} - z^{*2} - z^{*2} \end{aligned}$$

Problem 4 [Harder]
Solution

- Therefore,

$$
\begin{aligned}
f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\
&= y^* - x^{*2} - y^{*2} \\
&< z^* - x^{*2} - y^{*2} \\
&< z^* - 2z^{*2} \\
&= \min\{z^*, z^*\} - z^{*2} - z^{*2} \\
&= f(z^*, z^*),
\end{aligned}
$$

Problem 4 [Harder]
Solution

- Therefore,

$$\begin{aligned} f(x^*, y^*) &= \min\{x^*, y^*\} - x^{*2} - y^{*2} \\ &= y^* - x^{*2} - y^{*2} \\ &< z^* - x^{*2} - y^{*2} \\ &< z^* - 2z^{*2} \\ &= \min\{z^*, z^*\} - z^{*2} - z^{*2} \\ &= f(z^*, z^*), \end{aligned}$$

which is a contradiction since $f(x^*, y^*) \geq f(x, y)\ \forall (x, y) \in \mathbb{R}^2$.

Problem 4 [Harder]
Solution

- Finally, since we know $x^* = y^*$, the problem simplifies to

$$\max_x x - 2x^2.$$

Problem 4 [Harder]
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- Finally, since we know $x^* = y^*$, the problem simplifies to

$$\max_x x - 2x^2.$$

  - FOC: $\frac{\partial \tilde{f}(x^*)}{\partial x} = 1 - 4x^* = 0 \implies (x^*, y^*) = (\frac{1}{4}, \frac{1}{4})$.

Problem 4 [Harder]
Solution

- Finally, since we know $x^* = y^*$, the problem simplifies to

$$\max_x x - 2x^2.$$

  - FOC: $\frac{\partial \tilde{f}(x^*)}{\partial x} = 1 - 4x^* = 0 \implies (x^*, y^*) = (\frac{1}{4}, \frac{1}{4})$.
  - SOC: $\frac{\partial^2 \tilde{f}(x^*)}{x^2} = -4 < 0$, so the objective function is concave and we have found the solution.

□

Problem 5

Problem 5

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Problem 10

Find the optimal solution for the following program

$$\max_{x,y} x \quad \text{subject to} \quad x^3 + y^2 = 0.$$

Is the Lagrange approach appropriate?

Problem 5
Solution

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The **constraint qualification fails**:

- $(x, y) = (0, 0)$, a critical point of the constraint, belongs to the constraint set, so the procedure is not well defined and we cannot use the Lagrange Theorem.

Problem 5
Solution

The **constraint qualification fails**:

- $(x, y) = (0, 0)$, a critical point of the constraint, belongs to the constraint set, so the procedure is not well defined and we cannot use the Lagrange Theorem.

- To see this, note that $(0, 0)$ solves

$$Dh(x, y)' = \begin{pmatrix} 3x^2 \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

while it satisfies $h(0, 0) = 0$, where $h(x, y) = x^3 + y^2$ is the constraint function.

Problem 5

Solution

While the Lagrange method is not appropriate, it is easy to see that $(x^*, y^*) = (0, 0)$ is a solution.

Notice that the solution $(x^*, y^*) = (0, 0)$ is **not a critical point of the Lagrangian**. There is no solution to the FOC

$$1 - 3\lambda x^2 = 0;\ 2\lambda y = 0,\ x^3 + y^2 = 0$$

□

Problem 6

Problem 6

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Problem 10

Solve the following problem

$$\max_{x_1,x_2} x_1^2 x_2 \text{ subject to } 2x_1^2 + x_2^2 = 3.$$

Problem 6
Solution

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Problem 10

- Check the **constraint qualification**:

Problem 6
Solution

- Check the **constraint qualification**:

  It is **satisfied**. The only critical point of the constraint is $(x_1, x_2) = (0, 0)$, the only solution to

  $$Dh(x_1, x_2)' = \begin{pmatrix} 4x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  and it is not in the constraint set as it does not satisfy

  $$h(x_1, x_2) = 2x_1^2 + x_2^2 - 3 = 0.$$

Problem 6

Solution

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Problem 10

- Form the **Lagrangian**:

$$\mathcal{L}(x_1, x_2, \mu) = x_1^2 x_2 - \mu(2x_1^2 + x_2^2 - 3)$$

Problem 6
Solution

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Problem 10

- Form the **Lagrangian**:

$$\mathcal{L}(x_1, x_2, \mu) = x_1^2 x_2 - \mu(2x_1^2 + x_2^2 - 3)$$

- Take the **FOC**:

$[x_1]$ $$\frac{\partial \mathcal{L}(x_1, x_2, \mu)}{\partial x_1} = 2x_1 x_2 - 4\mu x_1 = 0$$

$[x_2]$ $$\frac{\partial \mathcal{L}(x_1, x_2, \mu)}{\partial x_2} = x_1^2 - 2\mu x_2 = 0$$

$[\mu]$ $$\frac{\partial \mathcal{L}(x_1, x_2, \mu)}{\partial \mu} = 2x_1^2 + x_2^2 - 3 = 0$$

Problem 6
Solution

- Solve to find the **critical points**: $(0, \sqrt{3}, 0)$, $(0, -\sqrt{3}, 0)$, $(1, 1, 1/2)$, $(-1, -1, -1/2)$, $(1, -1, -1/2)$, and $(-1, 1, 1/2)$.

details

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Problem 6
Solution

- Solve to find the **critical points**: $(0, \sqrt{3}, 0)$, $(0, -\sqrt{3}, 0)$, $(1, 1, 1/2)$, $(-1, -1, -1/2)$, $(1, -1, -1/2)$, and $(-1, 1, 1/2)$.

details

- Finally, **compact constraint set** $\implies \exists$ constrained max (and min). Must be among the critical points, so just plug into objective function and compare. The solution is:

$$(x_1^*, x_2^*) \in \{(1, 1), (-1, 1)\}.$$

details

□

Problem 7

Problem 7 [Harder]

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Problem 10

Solve the following problem when $a \in \left[\frac{1}{2}, \frac{3}{2}\right]$

$$\max_{x,y\geq 0} x^2 + y^2 \text{ subject to } ax + y = 1.$$

Problem 7 [Harder]
Solution

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Note that:

- The objective function and the Lagrangian are convex

Problem 7 [Harder]

Solution

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Note that:

- The objective function and the Lagrangian are convex $\implies$ FOC will identify a minimum.

Problem 7 [Harder]
Solution

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Note that:

- The objective function and the Lagrangian are convex $\implies$ FOC will identify a minimum.
- The k-level curves of the objective function are quarter-circles of radius $\sqrt{k}$ centered at the origin in $\mathbb{R}^2_+$:

$$x^2 + y^2 = k.$$

Problem 7 [Harder]

Solution

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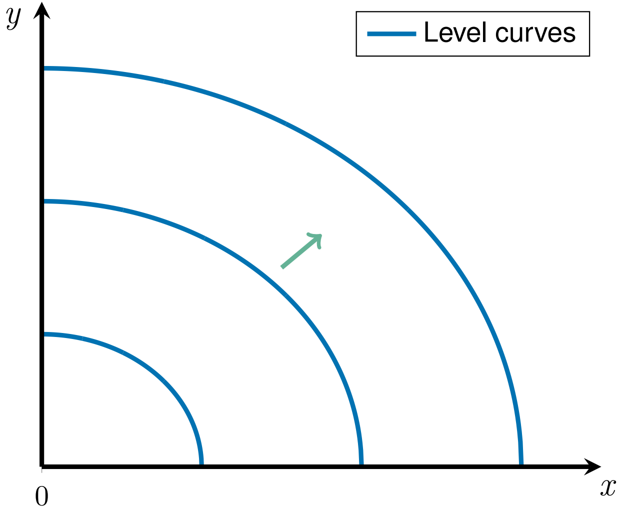

Problem 7 [Harder]
Solution

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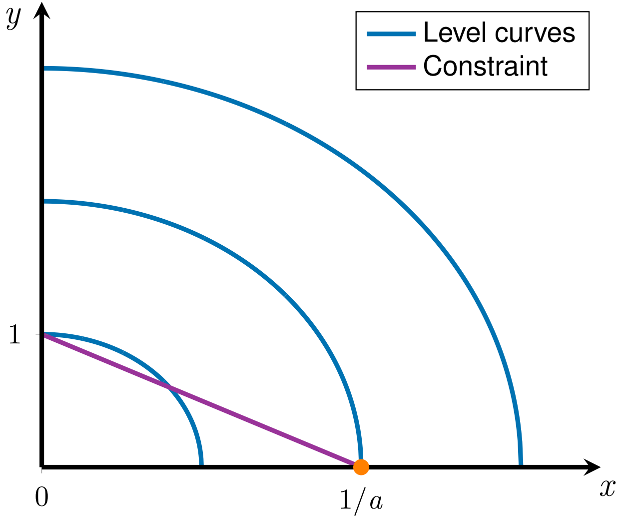

Problem 7 [Harder]

Solution

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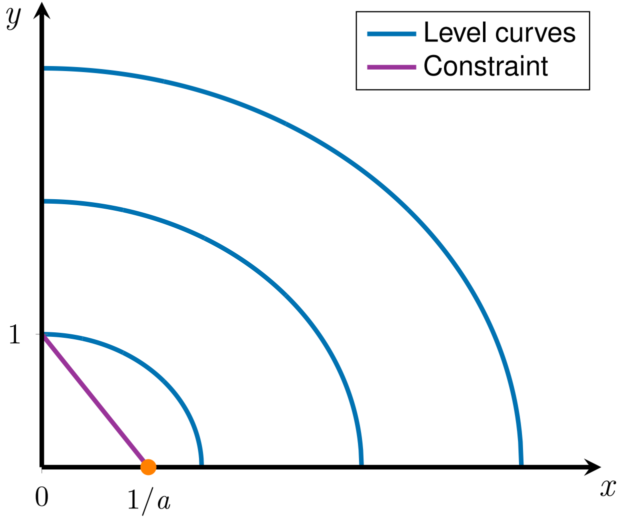

Problem 7 [Harder]
Solution

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- The solution is given by the corner (potentially both) that yields the higher value of the objective function:

$$f(0,1) = 1 \gtreqless \frac{1}{a^2} = f(\frac{1}{a}, 1)$$

$$\iff \quad a^2 \lesseqgtr 1.$$

Problem 7 [Harder]
Solution

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- The solution is given by the corner (potentially both) that yields the higher value of the objective function:

$$f(0,1) = 1 \gtreqless \frac{1}{a^2} = f(\frac{1}{a}, 1)$$

$$\iff a^2 \lesseqgtr 1.$$

- Thus, for $a \in \left[\frac{1}{2}, \frac{3}{2}\right]$, the solution to the constrained optimization problem is

$$(x^*, y^*) = \begin{cases} \left(\frac{1}{a}, 0\right) & \text{for } a \in \left[\frac{1}{2}, 1\right] \\ (0,1) & \text{for } a \in \left[1, \frac{2}{3}\right] \end{cases}. \quad \square$$

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Problem 8 [Harder]

Let X be a convex subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$ a concave function, $g : X \to \mathbb{R}^m$ a convex function, and a a vector in $\mathbb{R}^m$. Consider the following problem

$$\max_{x \in X} f(x) \text{ subject to } g(x) \leq a.$$

What is the Lagrangian for this problem? Prove that the Lagrangian is a concave function of the choice variable x on X.

Problem 8 [Harder]
Solution

The Lagrangian is

$$\mathcal{L}\left(x, \mu \mid a\right) = f\left(x\right) - \mu\left(g(x) - a\right),$$

where $\mu \in \mathbb{R}^m$ is a row vector of Lagrange multipliers.

Problem 8 [Harder]
Solution

Proof.

□

Problem 8 [Harder]

Solution

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Proof.

- Take any $x, x' \in X$.

$\square$

Problem 8 [Harder]

Solution

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Proof.

- Take any $x, x' \in X$. Then, $\forall\, t \in [0,1]$,

□

Problem 8 [Harder]
Solution

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Proof.

- Take any $x, x' \in X$. Then, $\forall\, t \in [0,1]$,

$$f(\cdot) \text{ concave } \implies f\left(tx + (1-t)\,x'\right) \geq tf(x) + (1-t)\,f(x')$$

□

Problem 8 [Harder]
Solution

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Proof.

- Take any $x, x' \in X$. Then, $\forall\, t \in [0,1]$,

$$f(\cdot) \text{ concave } \implies f\left(tx + (1-t)\,x'\right) \geq tf(x) + (1-t)\,f(x')$$

$$g(\cdot) \text{ convex } \implies -\mu g\left(tx + (1-t)\,x'\right) \geq -\mu\left(tg(x) + (1-t)\,g(x')\right)$$

□

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Proof.

- Take any $x, x' \in X$. Then, $\forall\, t \in [0,1]$,

$$f(\cdot) \text{ concave } \implies f\left(tx + (1-t)\,x'\right) \geq tf(x) + (1-t)\,f(x')$$

$$g(\cdot) \text{ convex } \implies -\mu g\left(tx + (1-t)\,x'\right) \geq -\mu\left(tg(x) + (1-t)\,g(x')\right)$$

$$\mathcal{L}\left(tx + (1-t)\,x', \mu \mid a\right) \geq t\Big[f(x) - \mu\big(g(x) - a\big)\Big] + (1-t)\,\Big[f(x') - \mu\big(g(x') - a\big)\Big]$$

$\square$

Problem 8 [Harder]

Solution

Problem 1
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Problem 10

Proof.

- Take any $x, x' \in X$. Then, $\forall\, t \in [0,1]$,

$$f(\cdot) \text{ concave } \implies f\left(tx + (1-t)\,x'\right) \geq tf(x) + (1-t)\,f(x')$$

$$g(\cdot) \text{ convex } \implies -\mu g\left(tx + (1-t)\,x'\right) \geq -\mu\left(tg(x) + (1-t)\,g(x')\right)$$

$$\mathcal{L}\left(tx + (1-t)\,x', \mu \mid a\right) \geq t\left[f(x) - \mu\left(g(x) - a\right)\right] + (1-t)\left[f(x') - \mu\left(g(x') - a\right)\right]$$

$$= t\mathcal{L}(x, \mu \mid a) + (1-t)\,\mathcal{L}(x', \mu \mid a).$$

□

Problem 9

Problem 9

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Consider the problem of maximizing xyz subject to $x + y + z \leq 1, x \geq 0, y \geq 0,$ and $z \geq 0$. Obviously, the three latter constraints do not bind, and we can concentrate only on the first constraint, $x + y + z \leq 1$. Find the solution and the Lagrange multiplier, and show how the optimal value would change if instead the constraint was changed to $x + y + z \leq 9/10$.

Problem 9

Solution

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- Constraint set has nonempty interior $\implies$ Slater condition satisfied.

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Problem 9
Solution

- Constraint set has nonempty interior $\implies$ Slater condition satisfied.
- Lagrangian:

$$\mathcal{L}(x, y, z, \mu) = xyz - \mu(x + y + z - 1).$$

Problem 9
Solution

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- FOC:

$$
\begin{aligned}
&[x] & y^*z^* - \mu = 0 \\
&[y] & x^*z^* - \mu = 0 \\
&[z] & x^*y^* - \mu = 0 \\
&[\mu] & x^* + y^* + z^* - 1 = 0
\end{aligned}
$$

Problem 9

Solution

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- From $[x]$ and $[y]$: $z^* \neq 0 \implies x^* = y^*$. From $[x]$ and $[z]$: $y^* \neq 0 \implies x^* = z^*$. Thus, $x^* = y^* = z^*$.

Problem 9
Solution

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Problem 10

- From [x] and [y]: $z^* \neq 0 \implies x^* = y^*$. From [$x$] and [$z$]: $y^* \neq 0 \implies x^* = z^*$. Thus, $x^* = y^* = z^*$.

- Plugging into [μ]: $x^* = y^* = z^* = \frac{1}{3} \implies \mu^* = \frac{1}{9}$.

Problem 9
Solution

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Problem 10

- From $[x]$ and $[y]$: $z^* \neq 0 \implies x^* = y^*$. From $[x]$ and $[z]$: $y^* \neq 0 \implies x^* = z^*$. Thus, $x^* = y^* = z^*$.
- Plugging into $[\mu]$: $x^* = y^* = z^* = \frac{1}{3} \implies \mu^* = \frac{1}{9}$.
- Optimal value: $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$.

Problem 9
Solution

- From $[x]$ and $[y]$: $z^* \neq 0 \implies x^* = y^*$. From $[x]$ and $[z]$: $y^* \neq 0 \implies x^* = z^*$. Thus, $x^* = y^* = z^*$.
- Plugging into $[\mu]$: $x^* = y^* = z^* = \frac{1}{3} \implies \mu^* = \frac{1}{9}$.
- Optimal value: $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$.
- If the constraint was changed to $x + y + z \leq \frac{9}{10}$, the solution would change to $3x^* = \frac{9}{10} \implies x^* = y^* = z^* = \frac{3}{10} \implies \mu^* = \frac{9}{100}$, and the optimal value would fall to $\left(\frac{3}{10}\right)^3 = \frac{27}{1000}$. □

Problem 10

Problem 10 [Harder]

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Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying:

$$f(x) = \begin{cases} u(x) & \text{if } g(x) \leq 0 \\ v(x) & \text{if } g(x) \geq 0 \end{cases}.$$

Further, suppose that: (i) $u(x) = v(x)$ if $g(x) = 0$; (ii) u and v are differentiable, strictly concave, and posses maximizer in $\mathbb{R}^n$; and (iii) $g(x)$ is differentiable and strictly convex. Carefully explain how you would solve the problem of maximizing f by choosing $x \in \mathbb{R}^n$.

Problem 10 [Harder]

Solution

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- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$.

Problem 10 [Harder]

Solution

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- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$.
  - Solve

$$\max_{x \in \mathbb{R}^n} u(x) \quad \text{subject to} \quad g(x) \leq 0.$$

Problem 10 [Harder]
Solution

Problem 1
Problem 2
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Problem 10

- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$.
  - Solve
  $$\max_{x \in \mathbb{R}^n} u(x) \ \text{ subject to } \ g(x) \leq 0.$$
  - Problem satisfies KKT conditions and admits solution since $u(\cdot)$ possesses maximizer in $\mathbb{R}^n$.

Problem 10 [Harder]
Solution

Problem 1
Problem 2
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Problem 10

- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$.
  - Solve
    $$\max_{x \in \mathbb{R}^n} u(x) \text{ subject to } g(x) \leq 0.$$
  - Problem satisfies KKT conditions and admits solution since $u(\cdot)$ possesses maximizer in $\mathbb{R}^n$.
  - Maximizers are the solutions to
    $$Du(x) - \lambda Dg(x) = 0, \lambda g(x) = 0, \lambda \geq 0, g(x) \leq 0.$$

Problem 10 [Harder]

Solution

- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$.
  - Solve

$$\max_{x \in \mathbb{R}^n} u(x) \text{ subject to } g(x) \leq 0.$$

  - Problem satisfies KKT conditions and admits solution since $u(\cdot)$ possesses maximizer in $\mathbb{R}^n$.
  - Maximizers are the solutions to

$$Du(x) - \lambda Dg(x) = 0, \lambda g(x) = 0, \lambda \geq 0, g(x) \leq 0.$$

  - Call a solution to this problem x_u^*.

Problem 10 [Harder]
Solution

Problem 1
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Problem 10

- If $\exists\, x \in \mathbb{R}^n : g(x) > 0$.

Problem 10 [Harder]

Solution

- If $\exists\, x \in \mathbb{R}^n : g(x) > 0$.
  - Solve

$$\max_{x\in\mathbb{R}^n} v(x) \text{ subject to } -g(x) \leq 0.$$

Problem 10 [Harder]
Solution

- If $\exists\, x \in \mathbb{R}^n : g(x) > 0$.
  - Solve
    $$\max_{x \in \mathbb{R}^n} v(x) \ \text{ subject to } \ -g(x) \leq 0.$$
  - Problem admits solution as $v(\cdot)$ possesses maximizer in $\mathbb{R}^n$. But since $-g(x)$ is strictly convex, does not satisfy conditions for KKT theorem. KKT FOC remain necessary conditions for max.

Problem 10 [Harder]

Solution

- If $\exists\, x \in \mathbb{R}^n : g(x) > 0$.
  - Solve

$$\max_{x \in \mathbb{R}^n} v(x) \text{ subject to } -g(x) \leq 0.$$

  - Problem admits solution as $v(\cdot)$ possesses maximizer in $\mathbb{R}^n$. But since $-g(x)$ is strictly convex, does not satisfy conditions for KKT theorem. KKT FOC remain necessary conditions for max.
  - Maximizers must be among the solutions to

$$Dv(x) + \lambda Dg(x) = 0, \lambda g(x) = 0, \lambda \geq 0, g(x) \geq 0.$$

Problem 10 [Harder]
Solution

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- If $\exists\, x \in \mathbb{R}^n : g(x) > 0$.
  - Solve
    $$\max_{x \in \mathbb{R}^n} v(x) \text{ subject to } -g(x) \leq 0.$$
  - Problem admits solution as $v(\cdot)$ possesses maximizer in $\mathbb{R}^n$. But since $-g(x)$ is strictly convex, does not satisfy conditions for KKT theorem. KKT FOC remain necessary conditions for max.
  - Maximizers must be among the solutions to
    $$Dv(x) + \lambda Dg(x) = 0, \lambda g(x) = 0, \lambda \geq 0, g(x) \geq 0.$$
  - Call a solution to this problem x_v^*.

Problem 10 [Harder]
Solution

Finally

- If $g(x) < 0 \forall x \in \mathbb{R}^n$, a solution is x_u^*.

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Problem 10 [Harder]

Solution

Finally

- If $g(x) < 0 \forall x \in \mathbb{R}^n$, a solution is x_u^*.
- If $g(x) > 0 \forall x \in \mathbb{R}^n$, a solution is x_v^*.

Problem 10 [Harder]
Solution

Problem 1
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Finally

- If $g(x) < 0 \forall\, x \in \mathbb{R}^n$, a solution is x_u^*.
- If $g(x) > 0 \forall\, x \in \mathbb{R}^n$, a solution is x_v^*.
- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$ and $\exists\, x \in \mathbb{R}^n : g(x) > 0$,

Problem 10 [Harder]
Solution

Finally

- If $g(x) < 0 \forall x \in \mathbb{R}^n$, a solution is x_u^*.
- If $g(x) > 0 \forall x \in \mathbb{R}^n$, a solution is x_v^*.
- If $\exists x \in \mathbb{R}^n : g(x) < 0$ and $\exists x \in \mathbb{R}^n : g(x) > 0$,
  - x_u^* is a solution if $u(x_u^*) \geq v(x_v^*)$; and

Problem 10 [Harder]

Solution

Finally

- If $g(x) < 0 \forall\, x \in \mathbb{R}^n$, a solution is x_u^*.
- If $g(x) > 0 \forall\, x \in \mathbb{R}^n$, a solution is x_v^*.
- If $\exists\, x \in \mathbb{R}^n : g(x) < 0$ and $\exists\, x \in \mathbb{R}^n : g(x) > 0$,
  - x_u^* is a solution if $u(x_u^*) \geq v(x_v^*)$; and
  - x_v^* is a solution if $u(x_u^*) \leq v(x_v^*)$.

$\square$

Appendix

Problem 1: Details

- $-6x^* + 6y^* = 0 \iff x^* = y^*$

return

Problem 1: Details

- $-6x^* + 6y^* = 0 \iff x^* = y^*$

- $4x^{*3} + 2x^* - 6x^* = 0 \iff x^*(x^*+1)(x^*-1) = 0 \iff x^* = 0$ or $x^* = -1$ or $x^* = 1$.

return

Problem 1: Details

Let $g(x) = x^2(x^2 - 2)$ and note that $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 4x(x+1)(x-1)$, so

- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} < 0 \; \forall \, x \in (-\infty, -1) \cup (0, 1)$,

return

Problem 1: Details

Let $g(x) = x^2(x^2 - 2)$ and note that $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 4x(x+1)(x-1)$, so

- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} < 0 \; \forall \, x \in (-\infty, -1) \cup (0, 1)$,
- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} > 0 \; \forall \, x \in (-1, 0) \cup (1, +\infty)$, and

return

Problem 1: Details

Let $g(x) = x^2(x^2 - 2)$ and note that $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 4x(x+1)(x-1)$, so

- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} < 0 \; \forall \, x \in (-\infty, -1) \cup (0, 1)$,
- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} > 0 \; \forall \, x \in (-1, 0) \cup (1, +\infty)$, and
- $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 0 \; \forall \, x \in \{-1, 0, 1\}$.

return

Problem 1: Details

Rewrite the system as

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  $\implies (x^*, y^*) \in \{(0,0), (0,1)\}$.
- $y^* = 0 \implies x^*({x^*}^2 - 1) = 0 \iff x^* = 0$ or $x^* = -1$ or $x^* = 1$
  $\implies (x^*, y^*) \in \{(-1,0), (1,0)\}$

Problem 1: Details

- $x^* \neq 0$ and $y^* \neq 0 \implies$

$$y^* + 3{x^*}^2 - 1 = 0$$

$$2y^* + {x^*}^2 - 1 = 0$$

$$\implies (x^*, y^*) \in \left\{ \left(-\frac{1}{\sqrt{5}}, \frac{2}{5}\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right) \right\}$$

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Problem 6: Details

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- If $x_1 = 0$:
  - From $[\mu]$: $x_2 = \pm\sqrt{3}$.
  - From $[x_2]$: $x_1 = 0$ and $x_2 \neq 0 \implies \mu = 0$.

So $(0, \sqrt{3}, 0)$ and $(0, -\sqrt{3}, 0)$ are critical points.

Problem 6: Details

- If $x_1 \neq 0$:

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return

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- If $x_1 \neq 0$:
  - From $[x_1]$: $\mu = \frac{x_2}{2}$.
  - Plugging into $[x_2]$: $x_1^2 = x_2^2$.
  - Plugging into $[\mu]$: $3x_1^2 = 3 \iff x_1 = \pm 1$.

So $(1, 1, 1/2)$, $(-1, -1, -1/2)$, $(1, -1, -1/2)$, and $(-1, 1, 1/2)$ are critical points.

return

Problem 6: Details

Evaluating the objective function $f(x_1, x_2) = x_1^2 x_2$ at the critical points:

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- $f(0, \sqrt{3}) = f(0, -\sqrt{3}) = 0$;

return

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Thus $(1, 1)$ and $(-1, 1)$ are maximizers.

return