EC400: SOFP

Class #1

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Problem 1

Problem 1

Show that the general quadratic form

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$

can be written as

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and find its unique symmetric representation.

Problem 1
Solution

Just multiply out:

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + 0\,x_2 & a_{12}x_1 + a_{22}x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$= a_{11}x_1^2 + 0\, x_1 x_2 + a_{12}x_1x_2 + a_{22}x_2^2$$

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$$= a_{11}x_1^2 + 0\,x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2.$$

Problem 1
Solution

The symmetric representation is

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

□

Problem 2

Problem 2

List all the principal minors of a general (3×3) matrix and denote which are the three leading principal minors.

Problem 2
Solution

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a generic 3×3 matrix.

Problem 2
Solution

The k^{th} order principal minors of $\boldsymbol{A}$ are the determinants of each k^{th}-order principal submatrix obtained by deleting $(3-k)$ columns and the corresponding rows of $\boldsymbol{A}$.

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The k^{th} order principal minors of $\boldsymbol{A}$ are the determinants of each k^{th}-order principal submatrix obtained by deleting $(3-k)$ columns and the corresponding rows of $\boldsymbol{A}$.

- 3^{rd} order:

$$\boldsymbol{A}_3 = \boldsymbol{A}$$

since we delete $3-3=0$ columns and rows. Therefore, the only 3^{rd}-order principal minor of $\boldsymbol{A}$ is

$$|\boldsymbol{A}_3| = |\boldsymbol{A}|.$$

Problem 2

Solution

- 2^{nd} order: Delete $3 - 2 = 1$ column and row.

Problem 2
Solution

- 2^{nd} order: Delete $3 - 2 = 1$ column and row.
  - Delete 1^{st} column and row:

$$\boldsymbol{A}_2^1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_2^1\right| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

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  - Delete 2^{nd} column and row:

$$\boldsymbol{A}_2^2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_2^2\right| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Problem 2
Solution

- Delete 3^{rd} column and row:

$$\boldsymbol{A}_2^3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_2^3\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

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- 1^{st} order: Delete $3 - 1 = 2$ columns and rows.

Problem 2

Solution

- Delete 3^{rd} column and row:

$$\boldsymbol{A}_2^3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_2^3\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- 1^{st} order: Delete $3 - 1 = 2$ columns and rows.
  - Delete 1^{st} and 2^{nd} columns and rows:

$$\boldsymbol{A}_1^{1,2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_1^{1,2}\right| = a_{33}$$

Problem 2
Solution

- Delete 1^{st} and 3^{rd} columns and rows:

$$\boldsymbol{A}_1^{1,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_1^{1,3}\right| = a_{22}$$

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- Delete 2^{nd} and 3^{rd} columns and rows:

$$\boldsymbol{A}_1^{2,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \left|\boldsymbol{A}_1^{2,3}\right| = a_{11}$$

Problem 2
Solution

The k^{th} order leading principal minors of $\boldsymbol{A}$ are the determinants of the k^{th}-order leading principal submatrices, obtained by deleting the **last** $(3-k)$ columns and rows of $\boldsymbol{A}$.

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$\implies$ The leading principal submatrices contain the first k elements of the diagonal. Thus,

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- $|L_1| = a_{11}$,

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$\implies$ The leading principal submatrices contain the first k elements of the diagonal. Thus,

- $|L_1| = a_{11}$,
- $|L_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, and

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- $|L_3| = |\boldsymbol{A}|$.

Problem 3

Problem 3

Determine the definiteness of the following symmetric matrices:

(a) $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$; (b) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$.

Problem 3
Solution

(a) $Q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c\, x_2^2$, so

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Solution

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- $c > 0 \implies Q(x) \geq 0 \; \forall x \neq 0$ and the matrix is **p.s.d.**

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- $c > 0 \implies Q(x) \geq 0 \; \forall x \neq 0$ and the matrix is **p.s.d.**
- $c < 0 \implies Q(x) \leq 0 \; \forall x \neq 0$ and the matrix is **n.s.d.**
- $c = 0 \implies Q(x) = 0 \; \forall x \neq 0$ and the matrix is both **p.s.d**. and **n.s.d.**

Problem 3
Solution

(b) The leading principal minors are:

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- $|L_1| = \det(2) = 2 > 0,$
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so the matrix is **p.d.** since both are strictly positive.

Problem 3
Solution

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- $|L_1| = \det(-3) = -3 < 0$,
- $|L_2| = \begin{vmatrix} -3 & 4 \\ 4 & -6 \end{vmatrix} = 18 - 16 = 2 > 0$,

so the matrix is **n.d.** since $|L_1|$ and $|L_2|$ alternate in sign accordingly.

Problem 3
Solution

(d) Note that

$$\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25 = -1 < 0,$$

which is the 2^{nd}-order principal minor that obtains by deleting row and column 1. Thus, the matrix is

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Solution

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which is the 2^{nd}-order principal minor that obtains by deleting row and column 1. Thus, the matrix is

- **not p.s.d.** since there is at least one strictly negative principal minor ($\implies$ **not p.d.** since $\exists x \neq 0 : Q(x) \leq 0$),

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- **not p.s.d.** since there is at least one strictly negative principal minor ($\implies$ **not p.d.** since $\exists x \neq 0 : Q(x) \leq 0$),
- **not n.s.d.** since at least one principal minor of even order is strictly negative ($\implies$ **not n.d.** since $\exists x \neq 0 : Q(x) \geq 0$).

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- **not n.s.d.** since at least one principal minor of even order is strictly negative ($\implies$ **not n.d.** since $\exists x \neq 0 : Q(x) \geq 0$).

That is, the matrix is **indefinite**. □

Problem 4

Problem 4 [Harder]

Consider the following quadratic form

$$Q(\mathbf{x}) = ax_1^2 + bx_2^2 + 2abx_1x_2.$$

For what values of the parameter values, a and b, is the quadratic form $Q(\mathbf{x})$ indefinite? Plot your answer in $\mathbb{R}^2$.

Problem 4 [Harder]

Solution

The symmetric representation of $Q(\mathbf{x})$ is given by the matrix

$$\boldsymbol{A} = \begin{pmatrix} a & ab \\ ab & b \end{pmatrix}$$

with principal minors

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Solution

The symmetric representation of $Q(\mathbf{x})$ is given by the matrix

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with principal minors

- 1^{st} order: a, b.
- 2^{nd} order: $|\boldsymbol{A}| = ab - (ab)^2 = ab(1 - ab)$.

Problem 4 [Harder]

Solution

Hence, $Q(\mathbf{x})$ is

Problem 4 [Harder]
Solution

Hence, $Q(\mathbf{x})$ is

- **p.s.d.** when

$$a \geq 0, b \geq 0, ab(1 - ab) \geq 0 \iff a \geq 0, b \geq 0, ab \leq 1.$$

Problem 4 [Harder]
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Hence, $Q(\mathbf{x})$ is

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- **n.s.d.** when

$$a \leq 0, b \leq 0, ab(1-ab) \geq 0 \iff a \leq 0, b \leq 0, ab \geq 1.$$

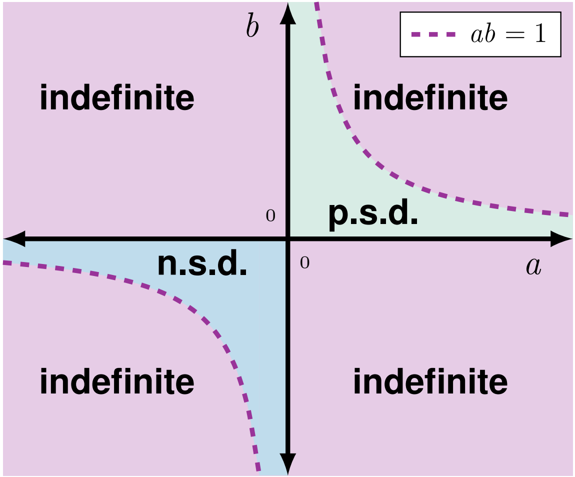
details

Problem 4 [Harder]
Solution

Hence, $Q(\mathbf{x})$ is

- **p.s.d.** when
$$a \geq 0, b \geq 0, ab(1-ab) \geq 0 \iff a \geq 0, b \geq 0, ab \leq 1.$$
- **n.s.d.** when
$$a \leq 0, b \leq 0, ab(1-ab) \geq 0 \iff a \leq 0, b \leq 0, ab \geq 1.$$
details
- **indefinite** when it is neither **p.s.d.** nor **n.s.d.**, i.e., in every other case:
$$\operatorname{sgn}(a) \neq \operatorname{sgn}(b) \text{ or } |b| > |1/a|.$$

Problem 4 [Harder]
Solution



Problem 5

Problem 5

Approximate e^x at $x = 0$ with a Taylor polynomial of order three and four. Then compute the values of these approximations at $h = 0.2$ and at $h = 1$ and compare with the actual values.

Problem 5
Solution

Class #1

EC400: SOFP

Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Problem 6
Problem 7
Problem 8
Problem 9
Problem 10

The 3^{rd}- and 4^{th}-order Taylor expansion of $f(x)$ around $x = a$ evaluated at $x = h$ are

Problem 5
Solution

The 3^{rd}- and 4^{th}-order Taylor expansion of $f(x)$ around $x = a$ evaluated at $x = h$ are

$$P_3(h \mid a) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{[3]}(a)}{3!}h^3$$

Problem 5

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Problem 5
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For $f(x) = e^x$, $\frac{\mathrm{d}^n f(x)}{\mathrm{d}\, x^n} = f(x) \; \forall n \in \mathbb{N}$.

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$$P_4(h \mid 0) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}$$

Problem 5
Solution

Finally, plugging in the values of h, we get

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h	$P_3(h \mid 0)$	$P_4(h \mid 0)$	$f(h)$
0.2	$1.221\overline{3}$	1.2214	1.221403
1	$2.\overline{6}$	$2.708\overline{3}$	2.718282

Problem 5
Solution

Finally, plugging in the values of h, we get

h	$P_3(h \mid 0)$	$P_4(h \mid 0)$	$f(h)$
0.2	$1.221\overline{3}$	1.2214	1.221403
1	$2.\overline{6}$	$2.708\overline{3}$	2.718282

Takeaway: Aim for Taylor expansions of low order but close to the approximation point. □

Problem 6

Problem 6

For each of the following functions on $\mathbb{R}$, determine whether they are quasiconcave, quasiconvex, both, or neither:

(a) e^x; (b) $\ln(x)$; (c) $x^3 - x$

Problem 6
Solution

(a) e^x is a strictly increasing function on $\mathbb{R}$. Therefore, it is **both** quasiconcave and quasiconvex.

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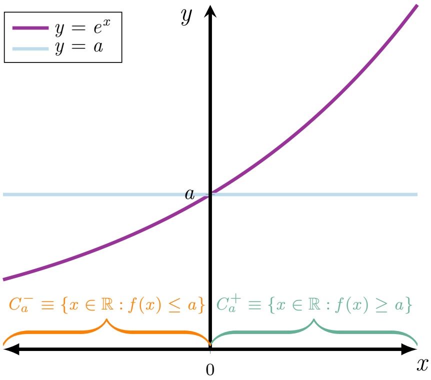

Problem 6
Solution

(b) By the same argument, $\ln(x)$ is **both** quasiconcave and quasiconvex.

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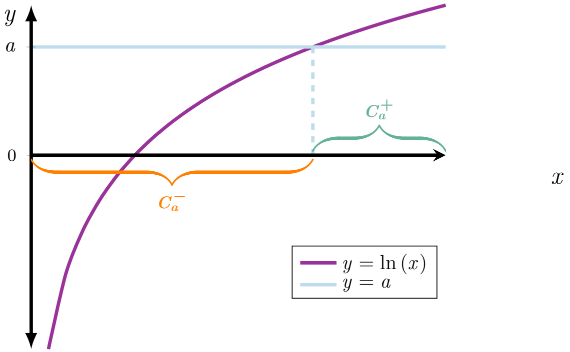
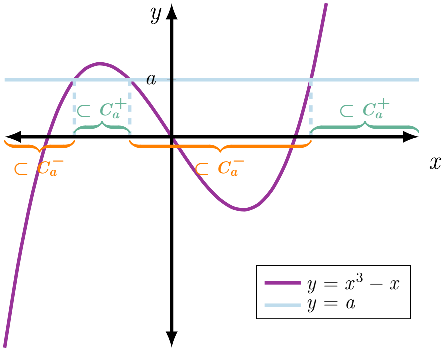


Problem 6
Solution

(c) $x^3 - x$ is **neither** quasiconcave nor quasiconvex since $\exists a \in \mathbb{R} : C_a^+$ is not convex and $\exists a \in \mathbb{R} : C_a^-$ is not convex.

Problem 6
Solution

(c) $x^3 - x$ is **neither** quasiconcave nor quasiconvex since $\exists a \in \mathbb{R} : C_a^+$ is not convex and $\exists a \in \mathbb{R} : C_a^-$ is not convex.



Problem 7

Problem 7 [Harder]

Let f be a function defined on a convex set U in $\mathbb{R}^n$. In lecture, we have shown that f is a quasiconcave function on U if and only if for all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0, 1]$

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

State the corresponding theorem for quasiconvex functions and prove it.

Problem 7 [Harder]
Solution

For the quasiconcave case, what does the statement

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$$

say?

Problem 7 `[Harder]`
Solution

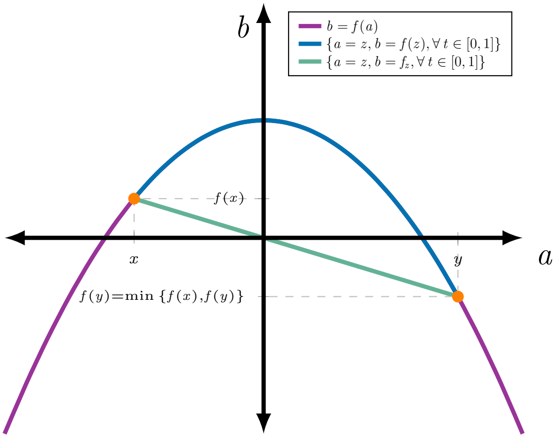
For the quasiconcave case, what does the statement

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$$

say?

Consider the following concave (and thus quasiconcave) function on $\mathbb{R}$, where for any $x, y \in \mathbb{R}$ and $t \in [0,1]$ we define $z \equiv tx + (1-t)y$ and $f_z \equiv tf(x) + (1-t)f(y)$.

Problem 7 [Harder]
Solution



Problem 7 [Harder]

Solution

Now, consider a convex (and thus quasiconvex) function on $\mathbb{R}$.

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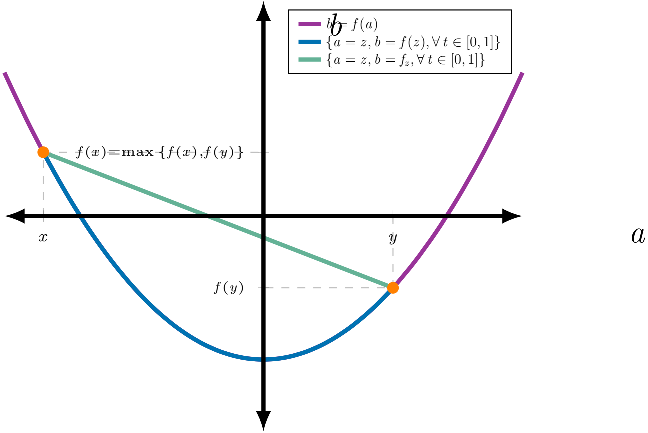

Problem 7 [Harder]
Solution

The corresponding theorem for quasiconvex functions is

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Theorem

Let f be a function defined on a convex set U in $\mathbb{R}^n$. Then, f is a quasiconvex function on U if and only if for all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0, 1]$

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

Problem 7 [Harder]

Solution

Proof.

Let U be a convex subset of $\mathbb{R}^n$.

Problem 7 [Harder]

Solution

Proof.

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Problem 7 [Harder]

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- B: "$\forall\, \mathbf{x}, \mathbf{y} \in U, \forall\, t \in [0,1], f(t\mathbf{x}, (1-t)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}$"

Problem 7 `[Harder]`
Solution

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We want to prove "A $\iff$ B", which can be broken into "A $\implies$ B" and "B $\implies$ A".

Problem 7 [Harder]
Solution

Proof: A $\implies$ B.

Let $f : U \to \mathbb{R}$ (where $U \subseteq \mathbb{R}^n$ is a convex set) be a quasiconvex function

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Let $f : U \to \mathbb{R}$ (where $U \subseteq \mathbb{R}^n$ is a convex set) be a quasiconvex function, i.e.,

$$C_k^- \equiv \{\mathbf{z} \in U : f(\mathbf{z}) \leq k\} \text{ is a convex set } \forall k \in \mathbb{R}.$$

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$f(\cdot)$ q-convex $\implies$ C_k^- convex $\implies$ $t\mathbf{x} + (1-t)\mathbf{y} \in C_k^- \; \forall\, t \in [0,1]$.

Problem 7 [Harder]

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Thus, by definition of C_k^-, $\forall\, t \in [0,1]$,

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Thus, by definition of C_k^-, $\forall\, t \in [0,1]$,

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This establishes that "A $\implies$ B".

Problem 7 [Harder]

Solution

Proof: B $\implies$ A.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $f : U \to \mathbb{R}$ be a function.

Problem 7 [Harder]
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Let $U \subseteq \mathbb{R}^n$ be a convex set and $f : U \to \mathbb{R}$ be a function.

Suppose that $\forall\, \mathbf{x}, \mathbf{y} \in U, \forall\, t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

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Thus, by our assumption, $\forall\, t \in [0, 1]$,

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$$\implies t\mathbf{x} + (1-t)\mathbf{y} \in C_k^-, \forall\, t \in [0,1] \iff C_k^- \text{ convex.}$$

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Since $k \in \mathbb{R}$ is arbitrary, C_k^- is convex $\forall\, k \in \mathbb{R} \iff f(\cdot)$ q-convex, establishing that "B $\implies$ A". $\square$

Problem 8

Problem 8

Prove that a weakly increasing transformation ($g : \mathbb{R} \to \mathbb{R}$ such that $g' \geq 0$) of a quasiconcave function is quasi-concave.

Problem 8

Solution

Proof.

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a q-concave function and consider any weakly increasing transformation $g : \mathbb{R} \to \mathbb{R}$ and the map $h = g \circ f$.

$\square$

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Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a q-concave function and consider any weakly increasing transformation $g : \mathbb{R} \to \mathbb{R}$ and the map $h = g \circ f$.

$f(\cdot)$ q-concave $\implies \forall \, \mathbf{x}, \mathbf{y} \in U, \forall \, t \in [0, 1]$,

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) \geq \min \{f(\mathbf{x}), f(\mathbf{y})\}.$$

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$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

$g'(\cdot) \geq 0 \implies \forall\, \mathbf{x}, \mathbf{y} \in U, \forall\, t \in [0,1]$,

$$g\big(f(t\mathbf{x} + (1-t)\mathbf{y})\big) \geq g\big(\min\{f(\mathbf{x}), f(\mathbf{y})\}\big) = \min\big\{g\big(f(\mathbf{x})\big), g\big(f(\mathbf{y})\big)\big\},$$

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i.e., $h(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{h(\mathbf{x}), h(\mathbf{y})\}$, so $h(\cdot)$ q-concave. □

Problem 9

Problem 9

A commonly used production or utility function on $\mathbb{R}^2_+$ is $f(x, y) = xy$. Check whether it is concave or convex using its Hessian. Then, check whether it is quasiconcave.

Problem 9
Solution

The Hessian is

$$D^2f(x,y) \equiv \begin{pmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Since the second order leading principal minor is

$$\det\left(D^2 f(x,y)\right) = 0 - 1 = -1 < 0,$$

the Hessian is indefinite and the function is **neither concave nor convex**.

Problem 9
Solution

To check quasiconcavity, consider the increasing transformation $g(x) = \ln(x)$.

Problem 9

Solution

To check quasiconcavity, consider the increasing transformation $g(x) = \ln(x)$. Note that

$$h(x, y) \equiv g\big(f(x, y)\big) = \ln(x) + \ln(y),$$

which is a sum of concave functions and thus concave $\implies h(x, y)$ quasiconcave.

Finally, since $g^{-1}(x) = e^x$ is also a strictly increasing transformation, the inverse mapping

$$f(x, y) = g^{-1}\big(h(x, y)\big)$$

must be **quasiconcave on** $\mathbb{R}^{\mathbf{2}}_{+}$.

note

Problem 10

Problem 10 [Harder]

Show that any continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$x \frac{\partial f(x)}{\partial x} \leq 0,$$

must be quasiconcave.

Problem 10 [Harder]
Solution

Proof.

The condition

$$x\frac{\partial f(x)}{\partial x} \leq 0$$

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- $x < 0 \implies \frac{\partial f(x)}{\partial x} \geq 0$
  ($f(\cdot)$ weakly increasing to the left of $x = 0$)

Problem 10 [Harder]

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- $x > 0 \implies \frac{\partial f(x)}{\partial x} \leq 0$
  ($f(\cdot)$ weakly decreasing to the right of $x = 0$)

Problem 10 [Harder]
Solution

Proof.

Therefore, the function must have a maximum at $x = 0$, i.e.

$$x = 0 \implies \frac{\partial f(x)}{\partial x} = 0.$$

We need to show that all upper-contour sets are convex (holds trivially for empty upper-contour sets).

Problem 10 [Harder]

Solution

Proof.

Take $k \leq f(0)$ and let

$$\underline{x}_k = \begin{cases} \min\{x \leq 0 : f(x) = k\} & \text{if finite} \\ -\infty & \text{otherwise} \end{cases}.$$

By continuity of $f(\cdot)$ and definition of $\underline{x}_k$, $\forall x \leq 0$,

$$f(x) \geq k \iff x \geq \underline{x}_k.$$

Problem 10 [Harder]

Solution

Proof.

Similarly, let

$$\overline{x}_k = \begin{cases} \max\{x \geq 0 : f(x) = k\} & \text{if finite} \\ +\infty & \text{otherwise} \end{cases}.$$

By continuity of $f(\cdot)$ and definition of $\overline{x}_k$, $\forall x \geq 0$,

$$f(x) \geq k \iff x \leq \overline{x}_k.$$

Problem 10 [Harder]

Solution

Proof.

Thus, $\forall k \leq f(0)$,

$$C_k^+ \equiv \{x \in \mathbb{R} : f(x) = k\} = [\underline{x}_k, \overline{x}_k],$$

which is an interval and thus convex $\implies f(\cdot)$ quasiconcave. □

Appendix

Problem 4: Details

Note that, given $a \leq 0$ and $b \leq 0$,

$$ab(1 - ab) \geq 0$$

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\iff \;& 1 - ab \geq 0 \\
\iff \;& ab \leq 1 \\
\iff \;& b \geq \frac{1}{a},
\end{aligned}
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ab(1-ab) &\geq 0 \\
\iff 1 - ab &\geq 0 \\
\iff ab &\leq 1 \\
\iff b &\geq \frac{1}{a},
\end{aligned}
$$

where the last equivalence follows by multiplying by $1/a \leq 0$ on both sides of the inequality, which flips the sign.

return

Problem 9: Note

It can be shown that $f(\cdot)$ is quasiconvex on $\mathbb{R}^2_-$ by a similar argument. Therefore, $f(\cdot)$ is neither quasiconvex nor quasiconcave on $\mathbb{R}^2$, but it is quasiconcave on $\mathbb{R}^2_+$ and quasiconvex on $\mathbb{R}^2_-$

return