EC400: SOFP

Class #1

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Show that the general quadratic form

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$

can be written as

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and find its unique symmetric representation.

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Just multiply out:

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + 0 x_2 & a_{12}x_1 + a_{22}x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Just multiply out:

$$(x_1 \quad x_2) \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_{11}x_1 + 0 x_2 \quad a_{12}x_1 + a_{22}x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= a_{11}x_1^2 + 0 x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2$$

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Just multiply out:

$$(x_1 \quad x_2) \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_{11}x_1 + 0 x_2 \quad a_{12}x_1 + a_{22}x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= a_{11}x_1^2 + 0 x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2.$$

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The symmetric representation is

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$



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List all the principal minors of a general (3×3) matrix and denote which are the three leading principal minors.

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Let

$$m{A} = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a generic 3×3 matrix.

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The $k^{\rm th}$ order principal minors of ${\bf A}$ are the determinants of each $k^{\rm th}$ -order principal submatrix obtained by deleting (3-k) columns and the corresponding rows of ${\bf A}$.

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The $k^{\rm th}$ order principal minors of ${\bf A}$ are the determinants of each $k^{\rm th}$ -order principal submatrix obtained by deleting (3-k) columns and the corresponding rows of ${\bf A}$.

• 3rd order:

$$A_3 = A$$

since we delete 3-3=0 columns and rows. Therefore, the only $3^{\rm rd}$ -order principal minor of ${\bf A}$ is

$$|\boldsymbol{A}_3|=|\boldsymbol{A}|.$$

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• 2^{nd} order: Delete 3-2=1 column and row.

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- 2^{nd} order: Delete 3-2=1 column and row.
 - Delete 1st column and row:

$$m{A}_2^1 = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies ig|m{A}_2^1ig| = egin{bmatrix} a_{22} & a_{23} \ a_{32} & a_{33} \ \end{pmatrix}$$

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- 2^{nd} order: Delete 3-2=1 column and row.
 - Delete 1st column and row:

$$m{A}_2^1 = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies ig|m{A}_2^1ig| = egin{bmatrix} a_{22} & a_{23} \ a_{32} & a_{33} \ \end{pmatrix}$$

Delete 2nd column and row:

$$m{A}_2^2 = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies ig|m{A}_2^2ig| = egin{bmatrix} a_{11} & a_{13} \ a_{31} & a_{33} \ \end{pmatrix}$$

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Delete 3rd column and row:

$$\mathbf{A}_{2}^{3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies |\mathbf{A}_{2}^{3}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

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Delete 3rd column and row:

$$\mathbf{A}_{2}^{3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies |\mathbf{A}_{2}^{3}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

• 1^{st} order: Delete 3-1=2 columns and rows.

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Delete 3rd column and row:

$$m{A}_2^3 = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies ig|m{A}_2^3ig| = egin{bmatrix} a_{11} & a_{12} \ a_{21} & a_{22} \ \end{pmatrix}$$

- 1st order: Delete 3-1=2 columns and rows.
 - Delete 1st and 2nd columns and rows:

$$m{A}_{1}^{1,2} = egin{array}{cccc} m{A}_{1}^{1,2} &= m{a}_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \ \end{pmatrix} \implies m{A}_{1}^{1,2} = a_{33}$$

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Delete 1st and 3rd columns and rows:

$$m{A}_1^{1,3} = egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies igg| m{A}_1^{1,3} igg| = a_{22}$$

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Delete 1st and 3rd columns and rows:

$$egin{aligned} m{A}_1^{1,3} &= egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \ \end{pmatrix} \implies igg|m{A}_1^{1,3}igg| = a_{22} \end{aligned}$$

Delete 2nd and 3rd columns and rows:

$$\mathbf{A}_{1}^{2,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{35} \end{pmatrix} \Longrightarrow \left| \mathbf{A}_{1}^{2,3} \right| = a_{11}$$

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The $k^{\rm th}$ order leading principal minors of \boldsymbol{A} are the determinants of the $k^{\rm th}$ -order leading principal submatrices, obtained by deleting the last (3-k) columns and rows of \boldsymbol{A} .

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The $k^{\rm th}$ order leading principal minors of \boldsymbol{A} are the determinants of the $k^{\rm th}$ -order leading principal submatrices, obtained by deleting the last (3-k) columns and rows of \boldsymbol{A} .



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The $k^{\rm th}$ order leading principal minors of \boldsymbol{A} are the determinants of the $k^{\rm th}$ -order leading principal submatrices, obtained by deleting the last (3-k) columns and rows of \boldsymbol{A} .

$$\bullet$$
 $|L_1| = a_{11},$

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The $k^{\rm th}$ order leading principal minors of \boldsymbol{A} are the determinants of the $k^{\rm th}$ -order leading principal submatrices, obtained by deleting the last (3-k) columns and rows of \boldsymbol{A} .

$$\bullet$$
 $|L_1| = a_{11},$

$$ullet$$
 $|L_2|=egin{vmatrix} a_{11}&a_{12}\ a_{21}&a_{22} \end{bmatrix}$, and



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The $k^{\rm th}$ order leading principal minors of ${\bf A}$ are the determinants of the $k^{\rm th}$ -order leading principal submatrices, obtained by deleting the last (3-k) columns and rows of ${\bf A}$.

$$\bullet$$
 $|L_1| = a_{11},$

$$ullet |L_2|=egin{array}{c|c} a_{11} & a_{12} \ a_{21} & a_{22} \ \end{array} ,$$
 and

$$|L_3| = |A|.$$



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Determine the definiteness of the following symmetric matrices:

(a)
$$\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$$
; (b) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$.

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(a)
$$Q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c x_2^2$$
, so

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(a)
$$Q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c x_2^2$$
, so

• $c > 0 \implies Q(x) \ge 0 \ \forall x \ne 0$ and the matrix is **p.s.d.**

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(a)
$$Q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c x_2^2$$
, so

• $c > 0 \implies Q(x) \ge 0 \ \forall x \ne 0$ and the matrix is **p.s.d.**

• $c < 0 \implies Q(x) \le 0 \ \forall x \ne 0$ and the matrix is **n.s.d.**

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(a)
$$Q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c x_2^2$$
, so

• $c > 0 \implies Q(x) \ge 0 \ \forall x \ne 0$ and the matrix is **p.s.d.**

• $c < 0 \implies Q(x) \le 0 \ \forall x \ne 0$ and the matrix is **n.s.d.**

• $c=0 \implies Q(x)=0 \ \forall x \neq 0$ and the matrix is both **p.s.d**. and **n.s.d**.

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(b) The leading principal minors are:

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(b) The leading principal minors are:

• $|L_1| = \det(2) = 2 > 0$,

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(b) The leading principal minors are:

•
$$|L_1| = \det(2) = 2 > 0$$
,

$$ullet |L_2| = egin{bmatrix} 2 & -1 \ -1 & 1 \end{bmatrix} = 2 - 1 = 1 > 0,$$

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(b) The leading principal minors are:

•
$$|L_1| = \det(2) = 2 > 0$$
,

$$ullet |L_2| = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2 - 1 = 1 > 0,$$

so the matrix is **p.d.** since both are strictly positive.

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(c) The leading principal minors are

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(c) The leading principal minors are

$$|L_1| = \det(-3) = -3 < 0,$$

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(c) The leading principal minors are

$$|L_1| = \det(-3) = -3 < 0,$$

$$|L_2| = \begin{vmatrix} -3 & 4 \\ 4 & -6 \end{vmatrix} = 18 - 16 = 2 > 0,$$

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The leading principal minors are

•
$$|L_1| = \det(-3) = -3 < 0$$
,

$$|L_2| = \begin{vmatrix} -3 & 4 \\ 4 & -6 \end{vmatrix} = 18 - 16 = 2 > 0,$$

so the matrix is **n.d.** since $|L_1|$ and $|L_2|$ alternate in sign accordingly.

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(d) Note that

$$\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25 = -1 < 0,$$

which is the $2^{\rm nd}$ -order principal minor that obtains by deleting row and column 1. Thus, the matrix is

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(d) Note that

$$\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25 = -1 < 0,$$

which is the $2^{\rm nd}$ -order principal minor that obtains by deleting row and column 1. Thus, the matrix is

• **not p.s.d.** since there is at least one strictly negative principal minor (\implies **not p.d.** since $\exists x \neq 0 : Q(x) \leq 0$),

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(d) Note that

$$\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25 = -1 < 0,$$

which is the $2^{\rm nd}$ -order principal minor that obtains by deleting row and column 1. Thus, the matrix is

- **not p.s.d.** since there is at least one strictly negative principal minor (\implies **not p.d.** since $\exists x \neq 0 : Q(x) \leq 0$),
- **not n.s.d.** since at least one principal minor of even order is strictly negative (\implies **not n.d.** since $\exists x \neq 0 : Q(x) \geq 0$).

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Note that

$$\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25 = -1 < 0,$$

which is the 2nd-order principal minor that obtains by deleting row and column 1. Thus, the matrix is

- **not p.s.d.** since there is at least one strictly negative principal minor (\Longrightarrow **not p.d.** since $\exists x \neq 0 : Q(x) < 0$),
- **not n.s.d.** since at least one principal minor of even order is strictly negative (\implies **not n.d.** since $\exists x \neq 0 : Q(x) > 0$).

That is, the matrix is **indefinite**.



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Consider the following quadratic form

$$Q(\mathbf{x}) = ax_1^2 + bx_2^2 + 2abx_1x_2.$$

For what values of the parameter values, a and b, is the quadratic form $Q(\mathbf{x})$ indefinite? Plot your answer in \mathbb{R}^2 .

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The symmetric representation of $Q(\mathbf{x})$ is given by the matrix

$$m{A} = egin{pmatrix} a & ab \ ab & b \end{pmatrix}$$

with principal minors

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The symmetric representation of $Q(\mathbf{x})$ is given by the matrix

$$oldsymbol{A} = egin{pmatrix} a & ab \ ab & b \end{pmatrix}$$

with principal minors

• 1^{st} order: a,b.

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Problem 1

The symmetric representation of $Q(\mathbf{x})$ is given by the matrix

$$oldsymbol{A} = egin{pmatrix} a & ab \ ab & b \end{pmatrix}$$

with principal minors

• 1^{st} order: a,b.

• 2^{nd} order: $|A| = ab - (ab)^2 = ab(1 - ab)$.

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Hence, $Q(\mathbf{x})$ is

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Hence, $Q(\mathbf{x})$ is

• p.s.d. when

$$a \ge 0, b \ge 0, ab(1 - ab) \ge 0 \iff a \ge 0, b \ge 0, ab \le 1.$$

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Hence, $Q(\mathbf{x})$ is

p.s.d. when

$$a \ge 0, b \ge 0, ab(1 - ab) \ge 0 \iff a \ge 0, b \ge 0, ab \le 1.$$

• n.s.d. when

$$a \le 0, b \le 0, ab(1 - ab) \ge 0 \iff a \le 0, b \le 0, ab \ge 1.$$

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Hence, $Q(\mathbf{x})$ is

p.s.d. when

$$a \ge 0, b \ge 0, ab(1 - ab) \ge 0 \iff a \ge 0, b \ge 0, ab \le 1.$$

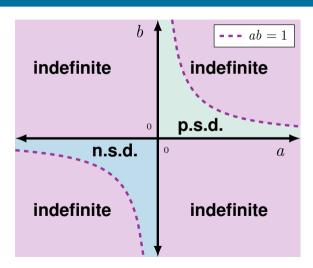
n.s.d. when

$$a \le 0, b \le 0, ab(1 - ab) \ge 0 \iff a \le 0, b \le 0, ab \ge 1.$$

• indefinite when it is neither p.s.d. nor n.s.d., i.e., in every other case:

$$\operatorname{sgn}(a) \neq \operatorname{sgn}(b) \text{ or } |b| > |1/a|.$$

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Approximate e^x at x=0 with a Taylor polynomial of order three and four. Then compute the values of these approximations at h=0.2 and at h=1 and compare with the actual values.

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The 3^{rd} - and 4^{th} -order Taylor expansion of f(x) around x=a evaluated at x=h are

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The 3^{rd} - and 4^{th} -order Taylor expansion of f(x) around x=a evaluated at x=h are

$$P_3(h \mid a) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{[3]}(a)}{3!}h^3$$

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The 3^{rd} - and 4^{th} -order Taylor expansion of f(x) around x=a evaluated at x=h are

$$P_3(h \mid a) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{[3]}(a)}{3!}h^3$$

$$P_4(h \mid a) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{[3]}(a)}{3!}h^3 + \frac{f^{[4]}(a)}{4!}h^4$$

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For
$$f(x) = e^x$$
, $\frac{\mathrm{d}^n f(x)}{\mathrm{d} x^n} = f(x) \ \forall n \in \mathbb{N}$.

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For
$$f(x) = e^x$$
, $\frac{d^n f(x)}{dx^n} = f(x) \ \forall n \in \mathbb{N}$. So,

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For
$$f(x) = e^x$$
, $\frac{d^n f(x)}{dx^n} = f(x) \ \forall n \in \mathbb{N}$. So,

$$P_3(h \mid a) = e^a + e^a h + \frac{e^a}{2!} h^2 + \frac{e^a}{3!} h^3$$

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For
$$f(x) = e^x$$
, $\frac{\mathrm{d}^n f(x)}{\mathrm{d} x^n} = f(x) \ \forall n \in \mathbb{N}$. So,

$$P_3(h \mid a) = e^a + e^a h + \frac{e^a}{2!} h^2 + \frac{e^a}{3!} h^3$$

$$P_4(h \mid a) = e^a + e^a h + \frac{e^a}{2!} h^2 + \frac{e^a}{3!} h^3 + \frac{e^a}{4!} h^4$$

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For a=0, these simplify to

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For a=0, these simplify to

$$P_3(h \mid 0) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6}$$

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For a = 0, these simplify to

$$P_3(h \mid 0) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6}$$

$$P_4(h \mid 0) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}$$

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Finally, plugging in the values of h, we get

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Finally, plugging in the values of h, we get

h	$P_3(h \mid 0)$	$P_4(h \mid 0)$	f(h)
0.2	$1.221\overline{3}$	1.2214	1.221403
1	$2.\overline{6}$	$2.708\overline{3}$	2.718282

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Finally, plugging in the values of h, we get

h	$P_3(h \mid 0)$	$P_4(h \mid 0)$	f(h)
0.2	$1.221\overline{3}$	1.2214	1.221403
1	$2.\overline{6}$	$2.708\overline{3}$	2.718282

Takeaway: Aim for Taylor expansions of low order but close to the approximation point.



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For each of the following functions on \mathbb{R} , determine whether they are quasiconcave, quasiconvex, both, or neither:

(a)
$$e^x$$
; (b) $\ln(x)$; (c) $x^3 - x$

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(a) e^x is a strictly increasing function on \mathbb{R} . Therefore, it is **both** quasiconcave and quasiconvex.

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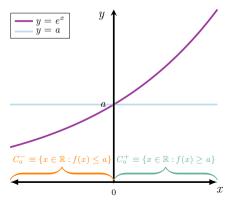
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(a) e^x is a strictly increasing function on \mathbb{R} . Therefore, it is **both** quasiconcave and quasiconvex.



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(b) By the same argument, $\ln(x)$ is **both** quasiconcave and quasiconvex.

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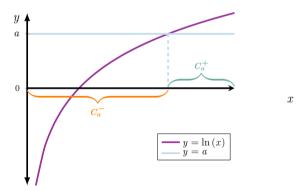
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(b) By the same argument, $\ln{(x)}$ is **both** quasiconcave and quasiconvex.



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(c) $x^3 - x$ is **neither** quasiconcave nor quasiconvex since $\exists a \in \mathbb{R} : C_a^+$ is not convex and $\exists a \in \mathbb{R} : C_a^-$ is not convex.

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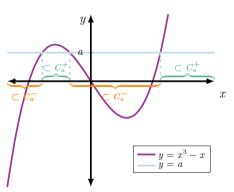
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(c) $x^3 - x$ is **neither** quasiconcave nor quasiconvex since $\exists a \in \mathbb{R} : C_a^+$ is not convex and $\exists a \in \mathbb{R} : C_a^-$ is not convex.



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Let f be a function defined on a convex set U in \mathbb{R}^n . In lecture, we have shown that f is a quasiconcave function on U if and only if for all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0,1]$

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

State the corresponding theorem for quasiconvex functions and prove it.

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For the quasiconcave case, what does the statement

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}\$$

say?

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For the quasiconcave case, what does the statement

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}\$$

say?

Consider the following concave (and thus quasiconcave) function on \mathbb{R} , where for any $x, y \in \mathbb{R}$ and $t \in [0,1]$ we define $z \equiv tx + (1-t)y$ and $f_z \equiv tf(x) + (1-t)f(y)$.

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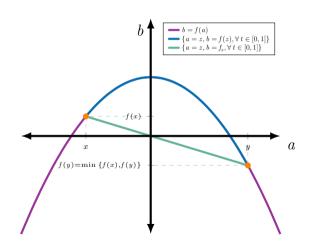
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Now, consider a convex (and thus quasiconvex) function on \mathbb{R} .

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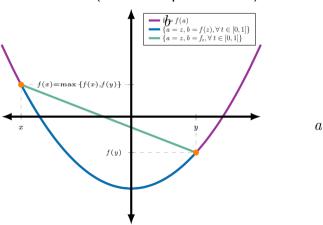
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Now, consider a convex (and thus quasiconvex) function on \mathbb{R} .



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The corresponding theorem for quasiconvex functions is

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The corresponding theorem for quasiconvex functions is

Theorem

Let f be a function defined on a convex set U in \mathbb{R}^n . Then, f is a quasiconvex function on U if and only if for all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0, 1]$

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

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Proof.

Let U be a convex subset of \mathbb{R}^n .

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Proof.

Let U be a convex subset of \mathbb{R}^n . Consider the statements

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Proof.

Let U be a convex subset of \mathbb{R}^n . Consider the statements

• A: " $f:U\to\mathbb{R}$ is a quasiconvex function"

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Proof.

Let U be a convex subset of \mathbb{R}^n . Consider the statements

- A: " $f:U\to\mathbb{R}$ is a quasiconvex function"
- B: " $\forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1], f(t\mathbf{x}, (1 t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}$ "

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Proof.

Let U be a convex subset of \mathbb{R}^n . Consider the statements

- A: " $f:U\to\mathbb{R}$ is a quasiconvex function"
- B: " $\forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1], f(t\mathbf{x}, (1 t)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}$ "

We want to prove "A \iff B", which can be broken into "A \implies B" and "B \implies A".

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Proof: $A \implies B$.

Let $f:U\to\mathbb{R}$ (where $U\subseteq\mathbb{R}^n$ is a convex set) be a quasiconvex function

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Proof: $A \implies B$.

Let $f:U\to\mathbb{R}$ (where $U\subseteq\mathbb{R}^n$ is a convex set) be a quasiconvex function, i.e.,

$$C_k^- \equiv \{ \mathbf{z} \in U : f(\mathbf{z}) \leq k \}$$
 is a convex set $\forall k \in \mathbb{R}$.

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Proof: $A \implies B$.

Let $f:U\to\mathbb{R}$ (where $U\subseteq\mathbb{R}^n$ is a convex set) be a quasiconvex function, i.e.,

$$C_k^- \equiv \{ \mathbf{z} \in U : f(\mathbf{z}) \leq k \}$$
 is a convex set $\forall k \in \mathbb{R}$.

Take any $\mathbf{x}, \mathbf{y} \in U$ and let $k = \max\{f(\mathbf{x}), f(\mathbf{y})\}$

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Proof: $A \implies B$.

Let $f:U\to\mathbb{R}$ (where $U\subseteq\mathbb{R}^n$ is a convex set) be a quasiconvex function, i.e.,

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 is a convex set $\forall k \in \mathbb{R}$.

Take any $\mathbf{x}, \mathbf{y} \in U$ and let $k = \max\{f(\mathbf{x}), f(\mathbf{y})\} \implies \mathbf{x}, \mathbf{y} \in C_k^-$.

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Proof: $A \implies B$.

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$$C_k^- \equiv \{ \mathbf{z} \in U : f(\mathbf{z}) \leq k \}$$
 is a convex set $\forall k \in \mathbb{R}$.

Take any $\mathbf{x}, \mathbf{y} \in U$ and let $k = \max\{f(\mathbf{x}), f(\mathbf{y})\} \implies \mathbf{x}, \mathbf{y} \in C_k^-$.

$$f(\cdot)$$
 q-convex $\implies C_k^-$ convex

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Proof: $A \implies B$.

Let $f:U\to\mathbb{R}$ (where $U\subseteq\mathbb{R}^n$ is a convex set) be a quasiconvex function, i.e.,

$$C_k^- \equiv \{ \mathbf{z} \in U : f(\mathbf{z}) \leq k \}$$
 is a convex set $\forall k \in \mathbb{R}$.

Take any $\mathbf{x}, \mathbf{y} \in U$ and let $k = \max\{f(\mathbf{x}), f(\mathbf{y})\} \implies \mathbf{x}, \mathbf{y} \in C_k^-$.

$$f(\cdot) \text{ q-convex} \implies C_k^- \text{ convex} \implies t\mathbf{x} + (1-t)\mathbf{y} \in C_k^- \ \forall \ t \in [0,1].$$

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Proof: $A \implies B$.

Thus, by definition of C_k^- ,

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Proof: $A \implies B$.

Thus, by definition of C_k^- , $\forall t \in [0,1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le k = \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

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Proof: $A \implies B$.

Thus, by definition of C_k^- , $\forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le k = \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

This establishes that "A \implies B".

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Proof: $B \implies A$.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $f: U \to \mathbb{R}$ be a function.

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Proof: $B \implies A$.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $f: U \to \mathbb{R}$ be a function.

Suppose that $\forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

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Proof: $B \implies A$.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $f: U \to \mathbb{R}$ be a function.

Suppose that $\forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

Let $k \in \mathbb{R}$ and take $\mathbf{x}, \mathbf{y} \in C_k^-$

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Proof: $B \implies A$.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $f: U \to \mathbb{R}$ be a function.

Suppose that $\forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

Let $k \in \mathbb{R}$ and take $\mathbf{x}, \mathbf{y} \in C_k^- \implies \max\{f(\mathbf{x}), f(\mathbf{y})\} \le k$.

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Proof: $B \implies A$.

Thus, by our assumption, $\forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\} \le k.$$



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Proof: $B \implies A$.

Thus, by our assumption, $\forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\} \le k.$$

$$\implies t\mathbf{x} + (1-t)\mathbf{y} \in C_k^-, \forall t \in [0,1]$$



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Proof: $B \implies A$.

Thus, by our assumption, $\forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\} \le k.$$

$$\implies t\mathbf{x} + (1-t)\mathbf{y} \in C_k^-, \forall t \in [0,1] \iff C_k^- \text{ convex}.$$



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Proof: $B \implies A$.

Thus, by our assumption, $\forall t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\} \le k.$$

$$\implies t\mathbf{x} + (1-t)\mathbf{y} \in C_k^-, \forall t \in [0,1] \iff C_k^- \text{ convex.}$$

Since $k \in \mathbb{R}$ is arbitrary, C_k^- is convex $\forall \, k \in \mathbb{R} \iff f(\cdot)$ q-convex, establishing that "B \implies A".

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Prove that a weakly increasing transformation $(g : \mathbb{R} \to \mathbb{R}$ such that $g' \ge 0$) of a quasiconcave function is quasi-concave.

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Proof.

Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be a q-concave function and consider any weakly increasing transformation $g:\mathbb{R}\to\mathbb{R}$ and the map $h=g\circ f$.





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Proof.

Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be a q-concave function and consider any weakly increasing transformation $g:\mathbb{R}\to\mathbb{R}$ and the map $h=g\circ f$.

$$f(\cdot)$$
 q-concave $\implies \forall \, \mathbf{x}, \mathbf{y} \in U, \forall \, t \in [0, 1],$

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$



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Proof.

Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be a q-concave function and consider any weakly increasing transformation $g:\mathbb{R}\to\mathbb{R}$ and the map $h=g\circ f$.

$$f(\cdot) \text{ q-concave } \implies \forall \, \mathbf{x}, \mathbf{y} \in \mathit{U}, \forall \, t \in [0,1],$$

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

$$g'(\cdot) \ge 0 \implies \forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0, 1],$$

$$g(f(t\mathbf{x} + (1-t)\mathbf{y})) \ge g(\min\{f(\mathbf{x}), f(\mathbf{y})\}) = \min\{g(f(\mathbf{x})), g(f(\mathbf{y}))\},$$



Proof.

Solution

Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be a q-concave function and consider any weakly

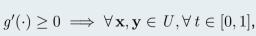
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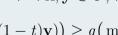
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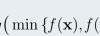
increasing transformation $g: \mathbb{R} \to \mathbb{R}$ and the map $h = g \circ f$.

 $f(\cdot)$ q-concave $\implies \forall \mathbf{x}, \mathbf{y} \in U, \forall t \in [0,1],$

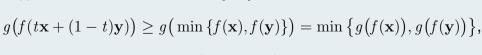


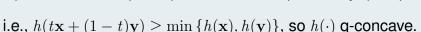












Class #1

Problem 9

A commonly used production or utility function on \mathbb{R}^2_+ is f(x,y)=xy. Check whether it is concave or convex using its Hessian. Then, check whether it is quasiconcave.

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The Hessian is

$$D^{2}f(x,y) \equiv \begin{pmatrix} \frac{\partial^{2}f(x,y)}{\partial x^{2}} & \frac{\partial^{2}f(x,y)}{\partial x\partial y} \\ \frac{\partial^{2}f(x,y)}{\partial y\partial x} & \frac{\partial^{2}f(x,y)}{\partial y^{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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The Hessian is

$$D^{2}f(x,y) \equiv \begin{pmatrix} \frac{\partial^{2}f(x,y)}{\partial x^{2}} & \frac{\partial^{2}f(x,y)}{\partial x\partial y} \\ \frac{\partial^{2}f(x,y)}{\partial y\partial x} & \frac{\partial^{2}f(x,y)}{\partial y^{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the second order leading principal minor is

$$\det (D^2 f(x,y)) = 0 - 1 = -1 < 0,$$

the Hessian is indefinite and the function is **neither concave nor convex**.

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To check quasiconcavity, consider the increasing transformation $g(x) = \ln(x)$.

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To check quasiconcavity, consider the increasing transformation $g(x) = \ln{(x)}$. Note that

$$h(x, y) \equiv g(f(x, y)) = \ln(x) + \ln(y),$$

which is a sum of concave functions and thus concave $\implies h(x,y)$ quasiconcave.

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Finally, since $g^{-1}(x)=e^x$ is also a strictly increasing transformation, the inverse mapping

$$f(x,y) = g^{-1}(h(x,y))$$

must be quasiconcave on \mathbb{R}^2_+ .



Problem 10 [Harder]

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Show that any continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$, satisfying

$$x\frac{\partial f(x)}{\partial x} \le 0,$$

must be quasiconcave.

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Proof.

The condition

$$x\frac{\partial f(x)}{\partial x} \le 0$$

implies that

Problem 10 [Harder]

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Proof.

The condition

$$x\frac{\partial f(x)}{\partial x} \le 0$$

implies that

•
$$x < 0 \implies \frac{\partial f(x)}{\partial x} \ge 0$$
 ($f(\cdot)$ weakly increasing to the left of $x = 0$)

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Proof.

The condition

$$x\frac{\partial f(x)}{\partial x} \le 0$$

implies that

- $x < 0 \implies \frac{\partial f(x)}{\partial x} \ge 0$ ($f(\cdot)$ weakly increasing to the left of x = 0)
- $x > 0 \implies \frac{\partial f(x)}{\partial x} \le 0$ ($f(\cdot)$ weakly decreasing to the right of x = 0)

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Proof.

Therefore, the function must have a maximum at x = 0, i.e.

$$x = 0 \implies \frac{\partial f(x)}{\partial x} = 0.$$

We need to show that all upper-contour sets are convex (holds trivially for empty upper-contour sets).

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Proof.

Take $k \leq f(0)$ and let

$$\underline{x}_k = \begin{cases} \min\left\{x \leq 0 : f(x) = k\right\} & \text{ if finite} \\ -\infty & \text{ otherwise} \end{cases}.$$

By continuity of $f(\cdot)$ and definition of \underline{x}_k , $\forall x \leq 0$,

$$f(x) \ge k \iff x \ge \underline{x}_k$$
.

Problem 10 [Harder]

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Proof.

Similarly, let

$$\overline{x}_k = \begin{cases} \max\left\{x \geq 0 : f(x) = k\right\} & \text{if finite} \\ +\infty & \text{otherwise} \end{cases}.$$

By continuity of $f(\cdot)$ and definition of \overline{x}_k , $\forall x \geq 0$,

$$f(x) \ge k \iff x \le \overline{x}_k.$$

Problem 10 [Harder]

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Proof.

Thus, $\forall k \leq f(0)$,

$$C_k^+ \equiv \{x \in \mathbb{R} : f(x) = k\} = [\underline{x}_k, \overline{x}_k],$$

which is an interval and thus convex $\implies f(\cdot)$ quasiconcave.



Appendix

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 ${\sf Appendix}$

Note that, given $a \leq 0$ and $b \leq 0$,

$$ab(1-ab) \ge 0$$

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Appendix

Note that, given $a \leq 0$ and $b \leq 0$,

$$ab(1-ab) \ge 0$$

$$\iff 1 - ab \ge 0$$

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Appendix

Note that, given $a \leq 0$ and $b \leq 0$,

$$ab(1-ab) \ge 0$$

$$\iff 1 - ab \ge 0$$

$$\iff ab \leq 1$$

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 ${\bf Appendix}$

Note that, given
$$a \leq 0$$
 and $b \leq 0$,

$$ab(1-ab) \ge 0$$

$$\iff 1 - ab \ge 0$$

$$\iff ab \le 1$$

$$\iff b \ge \frac{1}{a},$$

Class #1

Appendix

Note that, given $a \le 0$ and $b \le 0$,

ab(1-ab) > 0

 $) \geq 0$

 $\iff 1 - ab \ge 0$

 $\iff ab \le 1$

_ 1

 $\iff b \ge \frac{1}{a},$

where the last equivalence follows by multiplying by $1/a \le 0$ on both sides of the inequality, which flips the sign.

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Problem 9: Note

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Appendix

It can be shown that $f(\cdot)$ is quasiconvex on \mathbb{R}^2_- by a similar argument. Therefore, $f(\cdot)$ is neither quasiconvex nor quasiconcave on \mathbb{R}^2_+ but it is quasiconcave on \mathbb{R}^2_+ and quasiconvex on \mathbb{R}^2_-

