EC400: DPDE

Class #3

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Problem 3

Problem 3

Consider the investment problem from Problem Set 1: given initial value of capital K_0, a firm chooses investment path $\{I_t\}_{t=0}^{\infty}$ to maximize profits

$$\sum_{t=0}^{\infty} \beta^t \left[AK_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right]$$

subject to the capital law of motion

$$K_{t+1} = (1-\delta)K_t + I_t.$$

Problem 3
Question 1

1. Write down the firm's problem in a recursive form.

Problem 3

Question 1: Solution

- From the law of motion of capital: $I = K' - (1-\delta)K$.

Problem 3

Question 1: Solution

- From the law of motion of capital: $I = K' - (1-\delta)K$.

- The Bellman equation is

$$V(K) = \max_{K'} \left\{ AK^{\alpha} - K' + (1-\delta)K - \frac{\phi}{2}\frac{(K' - (1-\delta)K)^2}{K} + \beta V(K') \right\}$$

Problem 3
Question 2

2. Use the first-order and envelope conditions to derive the optimality condition for K_t.

Problem 3

Question 2: Solution

- FOC:

$$-1 - \phi \frac{K' - (1-\delta)K}{K} + \beta V'(K') = 0$$

$$\iff \quad -1 - \phi \frac{I}{K} + \beta V'(K') = 0$$

Problem 3
Question 2: Solution

- FOC:

$$-1-\phi\frac{K'-(1-\delta)K}{K}+\beta V'(K')=0$$

$$\iff -1-\phi\frac{I}{K}+\beta V'(K')=0$$

- Envelope condition:

$$V'(K)=\alpha AK^{\alpha-1}+(1-\delta)+\phi(1-\delta)\frac{I}{K}+\frac{\phi}{2}\frac{I^2}{K^2}$$

$$\implies V'(K')=\alpha AK'^{\alpha-1}+(1-\delta)\left[1+\phi\frac{I'}{K'}\right]+\frac{\phi}{2}\frac{I'^2}{K'^2}$$

Problem 3

Question 2: Solution

- Combining the FOC and envelope condition and rearranging:

$$1 + \phi \frac{I_t}{K_t} = \beta \left[\alpha A K_{t+1}^{\alpha-1} + \frac{\phi}{2} \frac{I_{t+1}^2}{K_{t+1}^2} + (1-\delta)\left(1 + \phi \frac{I_{t+1}}{K_{t+1}}\right) \right]$$

- Combining the FOC and envelope condition and rearranging:

$$1 + \phi \frac{I_t}{K_t} = \beta \left[\alpha A K_{t+1}^{\alpha-1} + \frac{\phi}{2} \frac{I_{t+1}^2}{K_{t+1}^2} + (1-\delta) \left(1 + \phi \frac{I_{t+1}}{K_{t+1}} \right) \right]$$

- Note this is the same optimality condition we obtained in PS1.

Problem 3
Question 3

3. Show that the solution to the Bellman equation exists and is unique.

Problem 3

Question 3: Solution

- Consider the metric space of bounded, continuous functions $f : X \to \mathbb{R}$ with $\rho(f, g) \equiv \max_{x \in X} |f(x) - g(x)|$, which can be shown to be a complete metric space.

Problem 3

Question 3: Solution

- Consider the metric space of bounded, continuous functions $f : X \to \mathbb{R}$ with $\rho(f, g) \equiv \max_{x \in X} |f(x) - g(x)|$, which can be shown to be a complete metric space.

- The Bellman operator

$$\mathrm{T} f \equiv \max_{B' \in \Gamma(B)} \{u(B, B') + \beta f(B')\},$$

with $u(\cdot, \cdot)$ bounded and continuous, and $\Gamma(\cdot)$ compact-valued and continuous, maps this space into itself.

Problem 3

Question 3: Solution

- The Bellman operator can be shown to be a contraction mapping.

Problem 3
Question 3: Solution

- The Bellman operator can be shown to be a contraction mapping.
- The contraction mapping theorem ensures the solution, a fixed point of T, exists and is unique.

Problem 3

Question 4

4. Rewrite the Bellman equation for an arbitrary period length of Δ.

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Question 4: Solution

- Note that the investment cost I_t and the adjustment cost $\frac{\phi}{2}\frac{I_t^2}{K_t}$ are flows.

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Question 4: Solution

- Note that the investment cost I_t and the adjustment cost $\frac{\phi}{2}\frac{I_t^2}{K_t}$ are flows.

- The problem for an arbitrary length of period Δ is

$$V(K_t) = \max_{I_t} \left\{ AK_t^{\alpha}\Delta - I_t\Delta - \frac{\phi}{2}\frac{I_t^2}{K_t}\Delta + e^{-\rho\Delta} V(K_{t+\Delta}) \right\}$$

$$\text{s.t. } K_{t+\Delta} = (1-\delta\Delta)K_t + I_t\Delta$$

Problem 3
Question 5

5. Derive the HJB equation taking the limit $\Delta \to 0$.

Problem 3

Question 5: Solution

- Using $e^{-\rho\Delta} \approx 1 - \rho\Delta$, rearranging, and dividing by Δ on both sides:

$$0 = \max_{I_t} \left\{ AK_t^{\alpha}\Delta - I_t\Delta - \frac{\phi}{2}\frac{I_t^2}{K_t}\Delta + (1-\rho\Delta)V(K_{t+\Delta}) - V(K_t) \right\}$$

$$\iff$$

$$\rho V(K_{t+\Delta}) = \max_{I_t} \left\{ AK_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t} + \frac{V(K_{t+\Delta}) - V(K_t)}{K_{t+\Delta} - K_t}\frac{K_{t+\Delta} - K_t}{\Delta} \right\}$$

Problem 3

Question 5: Solution

- Using $e^{-\rho\Delta} \approx 1 - \rho\Delta$, rearranging, and dividing by Δ on both sides:

$$0 = \max_{I_t} \left\{ AK_t^{\alpha}\Delta - I_t\Delta - \frac{\phi}{2}\frac{I_t^2}{K_t}\Delta + (1-\rho\Delta)V(K_{t+\Delta}) - V(K_t) \right\}$$

$$\iff$$

$$\rho V(K_{t+\Delta}) = \max_{I_t} \left\{ AK_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t} + \frac{V(K_{t+\Delta}) - V(K_t)}{K_{t+\Delta} - K_t}\frac{K_{t+\Delta} - K_t}{\Delta} \right\}$$

- Taking the limit as $\Delta \to 0$ on both sides:

$$\rho V(K_t) = \max_{I_t} \left\{ AK_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t} + V'(K_t)\dot{K}_t \right\}$$

Problem 3

Question 5: Solution

- From the law of motion of capital:

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$$

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$$\dot{K}_t = I_t - \delta K_t$$

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Question 5: Solution

- From the law of motion of capital:
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- Taking the limit as $\Delta \to 0$ on both sides:
$$\dot{K}_t = I_t - \delta K_t$$

- Finally, plugging into the expression from the previous slide, we obtain the HJB equation:
$$\rho V(K) = \max_I \left\{ AK^\alpha - I - \frac{\phi}{2}\frac{I^2}{K} + V'(K)\left(I - \delta K\right) \right\}$$