EC400: DPDE

Class #2

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Contents

Problem 1

Problem 1

The evolution of x_t is described by a linear system

$$x_{t+1} = Ax_t, \qquad A = \begin{pmatrix} 1 & -0.5 \\ -1 & 1.5 \end{pmatrix}.$$

Problem 1
Question 1

1. Find the steady state of the system. Is it a source, a sink, or a saddle point?

Problem 1

Question 1: Solution

- **Steady state:**

$$A\bar{x} = \bar{x}$$

$$\iff (A - I)\,\bar{x} = 0$$

$$\iff \begin{pmatrix} 0 & -0.5 \\ -1 & 0.5 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Problem 1

Question 1: Solution

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- The only solution is $\bar{x} = 0$.

Problem 1

Question 1: Solution

- **eigenvalues:** An eigenvalue λ solves

$$|A - \lambda I| = 0$$

$$\iff \begin{vmatrix} 1-\lambda & -0.5 \\ -1 & 1.5-\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 - 2.5\lambda + 1 = 0$$

Problem 1
Question 1: Solution

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- Solving, $\boxed{\lambda_1 = 2}$ and $\boxed{\lambda_2 = 0.5}$.

Problem 1

Question 1: Solution

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- Solving, $\boxed{\lambda_1 = 2}$ and $\boxed{\lambda_2 = 0.5}$.
- $|\lambda_1| > 1$ and $|\lambda_2| < 1 \implies$ SS is a **saddle point.**

Problem 1

Question 2

2. Draw the iso-lines and show with arrows the directions of the trajectories.

Problem 1
Question 2: Solution

- From the first equation:
  $x_{1,t+1} = x_{1,t} - 0.5x_{2,t} \iff \boxed{\Delta x_{1,t+1} = -0.5x_{2,t}}$.

Problem 1
Question 2: Solution

- From the first equation:
  $x_{1,t+1} = x_{1,t} - 0.5x_{2,t} \iff \boxed{\Delta x_{1,t+1} = -0.5x_{2,t}}$.
- From the second equation:
  $x_{2,t+1} = -x_{1,t} + 1.5x_{2,t} \iff \boxed{\Delta x_{2,t+1} = -x_{1,t} + 0.5x_{2,t}}$.

Problem 1

Question 2: Solution

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- Thus, the iso-lines are $\Delta x_{1,t+1} = 0 \iff \boxed{x_{2,t} = 0}$ and $\Delta x_{2,t+1} = 0 \iff \boxed{x_{2,t} = 2x_{1,t}}$.

Problem 1
Question 2: Solution

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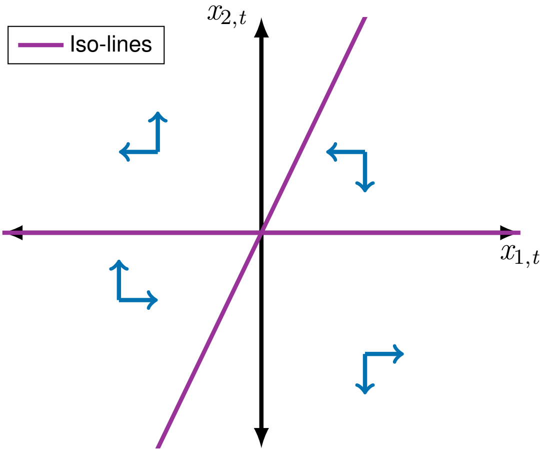
- Thus, the iso-lines are $\Delta x_{1,t+1} = 0 \iff \boxed{x_{2,t} = 0}$ and $\Delta x_{2,t+1} = 0 \iff \boxed{x_{2,t} = 2x_{1,t}}$.

- Moreover, $\boxed{\Delta x_{1,t+1} \gtrless 0 \iff x_{2,t} \lessgtr 0}$ and $\boxed{\Delta x_{2,t+1} \gtrless 0 \iff x_{2,t} \gtrless 2x_{1,t}}$

Problem 1
Question 2: Solution



Problem 1
Question 3

3. Decompose matrix A and show the eigenvectors on the phase diagram.

Problem 1

Question 3: Solution

- If λ is an eigenvalue, its eigenvector v solves $(A - \lambda I)\, v = 0$. Thus

Problem 1
Question 3: Solution

- If λ is an eigenvalue, its eigenvector v solves $(A - \lambda I)\, v = 0$. Thus
  - For $\lambda_1 = 2$:

$$\begin{pmatrix} -1 & -0.5 \\ -1 & -0.5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\iff \quad v_2 = -2v_1,$$

So $\boxed{v^1 = \begin{pmatrix} 1 & -2 \end{pmatrix}'}$.

Problem 1
Question 3: Solution

- For $\lambda_1 = 0.5$:

$$\begin{pmatrix} 0.5 & -0.5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\iff \quad v_2 = v_1,$$

So $\boxed{v^2 = \begin{pmatrix} 1 & 1 \end{pmatrix}'}$.

Problem 1

Question 3: Solution

- The matrix of eigenvectors is then

$$Q = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

with inverse

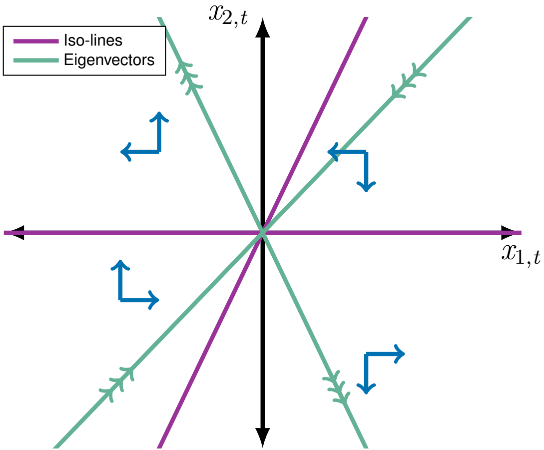
$$Q^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

- Finally, the decomposition of matrix A is

$$A = Q\Lambda Q^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Problem 1

Question 3: Solution



Problem 1
Question 4

4. Consider $x_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}'$, compute y_0 of the corresponding decoupled system, and show dynamics for periods $t = 1$ and $t = 2$ on the phase diagram.

Problem 1

Question 4: Solution

- Let $y_t = Q^{-1}x_t \implies y_{t+1} = \Lambda y_t$. Then,

Problem 1

Question 4: Solution

- Let $y_t = Q^{-1}x_t \implies y_{t+1} = \Lambda y_t$. Then,
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Problem 1
Question 4: Solution

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  - $y_1 = \Lambda y_0 = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.

Problem 1
Question 4: Solution

- Let $y_t = Q^{-1}x_t \implies y_{t+1} = \Lambda y_t$. Then,
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  - $y_1 = \Lambda y_0 = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.
  - $x_1 = Qy_1 = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Problem 1
Question 4: Solution

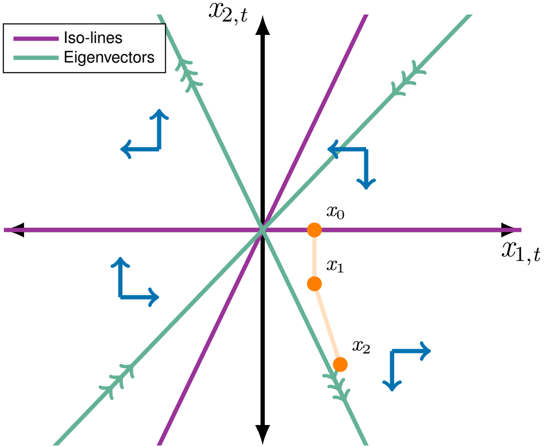
- $$y_2 = \Lambda y_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{6} \end{pmatrix}.$$

Problem 1
Question 4: Solution

- $y_2 = \Lambda y_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{6} \end{pmatrix}$.

- $x_2 = Qy_2 = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}$.

Problem 1
Question 4: Solution



Problem 3

Problem 3

Consider the investment problem from Problem Set 1: given initial value of capital K_0, a firm chooses investment path $\{I_t\}_{t=0}^{\infty}$ to maximize profits

$$\sum_{t=0}^{\infty} \beta^t \left[A K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right]$$

subject to the capital law of motion

$$K_{t+1} = (1-\delta) K_t + I_t.$$

Assume that $\delta = 0$ and define $Q_t \equiv \beta^{-t} \lambda_t$, where λ_t is the Lagrange multiplier from the optimization problem.

Problem 3
Question 1

1. Write down the (non-linear) system of equations for K_t and Q_t.

Problem 3

Question 1: Solution

- From PS1 and using $\boxed{\delta = 0}$:

$$[I_t] \qquad -\beta^t\left(1 + \phi\frac{I_t}{K_t}\right) + \lambda_t = 0$$

$$\iff 1 + \phi\frac{I_t}{K_t} = Q_t$$

$$[K_{t+1}] \qquad -\lambda_t + \beta^{t+1}\left(\alpha A K_{t+1}^{\alpha-1} + \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\right) + \lambda_{t+1} = 0$$

$$\iff \beta\left[\alpha A K_{t+1}^{\alpha-1} + \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2} + Q_{t+1}\right] = Q_t$$

Problem 3
Question 1: Solution

- From the capital law of motion: $I_t = K_{t+1} - K_t$. Substituting into the FOC:

$$[I_t] \qquad 1 + \phi \frac{K_{t+1}}{K_t} - \phi = Q_t$$

$$[K_{t+1}] \qquad \beta \left[\alpha A K_{t+1}^{\alpha-1} + \frac{\phi}{2} \frac{(K_{t+2} - K_{t+1})^2}{K_{t+1}^2} + Q_{t+1} \right] = Q_t$$

Problem 3
Question 2

2. Determine the state variables and the control variables in this system.

Problem 3

Question 2: Solution

- K_t is a state variable.

Problem 3

Question 2: Solution

- K_t is a state variable.
- Q_t is a control variable.

Problem 3
Question 3

3. Solve for the steady state.

Problem 3
Question 3: Solution

- In a SS $K_t = \bar{K} \; \forall \, t$ and $Q_t = \bar{Q} \; \forall \, t$.

Problem 3
Question 3: Solution

- In a SS $K_t = \bar{K} \; \forall \, t$ and $Q_t = \bar{Q} \; \forall \, t$.
- Imposing SS in $[I_t]$: $\bar{Q} = 1 + \phi \frac{\bar{K}}{\bar{K}} - \phi \iff \boxed{\bar{Q} = 1}$.

Problem 3

Question 3: Solution

- In a SS $K_t = \bar{K} \; \forall \, t$ and $Q_t = \bar{Q} \; \forall \, t$.
- Imposing SS in $[I_t]$: $\bar{Q} = 1 + \phi \frac{\bar{K}}{\bar{K}} - \phi \iff \boxed{\bar{Q} = 1}$.
- Imposing SS in $[K_{t+1}]$ and using $\bar{Q} = 1$:

$$1 = \beta \left[\alpha A \bar{K}^{\alpha - 1} + \frac{\phi}{2} \frac{\left(\bar{K} - \bar{K}\right)^2}{\bar{K}^2} + 1 \right] \iff \boxed{\bar{K} = \left(\frac{\beta \alpha A}{1 - \beta} \right)^{\frac{1}{1-\alpha}}}.$$

Problem 3
Question 4

4. Log-linearize the equations for K_t and Q_t around the steady state. Denote the log deviations with small letters k_t and q_t.

Problem 3

Question 4: Solution

- Let $q_t \equiv \ln\left(\frac{Q_t}{\bar{Q}}\right)$ and note that

$$Q_t = \bar{Q}\,\frac{Q_t}{\bar{Q}} = \bar{Q}\,e^{\ln\left(\frac{Q_t}{\bar{Q}}\right)} = \bar{Q}\,e^{q_t} \approx \bar{Q}\,(1+q_t) = (1+q_t)\,.$$

Problem 3

Question 4: Solution

- Similarly, let $k_t \equiv \ln\left(\frac{K_t}{\bar{K}}\right)$ and note that

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Problem 3
Question 4: Solution

- Similarly, let $k_t \equiv \ln\left(\frac{K_t}{\bar{K}}\right)$ and note that
  - $K_t = \bar{K}\, e^{k_t} \approx \bar{K}\,(1 + k_t)$.
  - $\frac{K_{t+1}}{K_t} = \frac{\bar{K}\, e^{k_{t+1}}}{\bar{K}\, e^{k_t}} = e^{k_{t+1} - k_t} \approx 1 + k_{t+1} - k_t$.

Problem 3
Question 4: Solution

- Similarly, let $k_t \equiv \ln\left(\frac{K_t}{\bar{K}}\right)$ and note that
  - $K_t = \bar{K}\, e^{k_t} \approx \bar{K}\,(1 + k_t)$.
  - $\frac{K_{t+1}}{K_t} = \frac{\bar{K}\, e^{k_{t+1}}}{\bar{K}\, e^{k_t}} = e^{k_{t+1} - k_t} \approx 1 + k_{t+1} - k_t$.
  - $\left(\frac{K_{t+2} - K_{t+1}}{K_{t+1}}\right)^2 = \left(\frac{\bar{K}\, e^{k_{t+2}}}{\bar{K}\, e^{k_{t+1}}} - 1\right)^2 = \left(e^{k_{t+2} - k_{t+1}} - 1\right)^2$ $\approx (1 + k_{t+2} - k_{t+1} - 1)^2 = (k_{t+2} - k_{t+1})^2$.

Problem 3

Question 4: Solution

- Similarly, let $k_t \equiv \ln\left(\frac{K_t}{\bar{K}}\right)$ and note that
  - $K_t = \bar{K}\,e^{k_t} \approx \bar{K}\,(1 + k_t)$.
  - $\frac{K_{t+1}}{K_t} = \frac{\bar{K}\,e^{k_{t+1}}}{\bar{K}\,e^{k_t}} = e^{k_{t+1}-k_t} \approx 1 + k_{t+1} - k_t$.
  - $\left(\frac{K_{t+2}-K_{t+1}}{K_{t+1}}\right)^2 = \left(\frac{\bar{K}\,e^{k_{t+2}}}{\bar{K}\,e^{k_{t+1}}} - 1\right)^2 = \left(e^{k_{t+2}-k_{t+1}} - 1\right)^2$ $\approx (1 + k_{t+2} - k_{t+1} - 1)^2 = (k_{t+2} - k_{t+1})^2$.
  - $\alpha A K_{t+1}^{\alpha-1} = \alpha A \left(\bar{K}\,e^{k_{t+1}}\right)^{\alpha-1} = \alpha A \bar{K}^{\alpha-1} e^{(\alpha-1)k_{t+1}}$ $\approx \alpha A \frac{1-\beta}{\beta\alpha A}\Big(1 + (\alpha-1)k_{t+1}\Big) = \frac{1-\beta}{\beta} - \kappa\, k_{t+1}$, where $\kappa \equiv \frac{(1-\alpha)(1-\beta)}{\beta}$.

Problem 3
Question 4: Solution

- Plugging these approximations into the non-linear system:

$$[I_t] \qquad 1 + q_t = 1 + \phi\,(1 + k_{t+1} - k_t) - \phi$$

$$\iff k_{t+1} = k_t + \frac{1}{\phi} q_t$$

$$[K_{t+1}] \qquad 1 + q_t = \beta\left[\frac{1-\beta}{\beta} - \kappa\, k_{t+1} + \frac{\phi}{2}\left(k_{t+2} - k_{t+1}\right)^2 + 1 + q_{t+1}\right]$$

Problem 3

Question 4: Solution

- Simplifying $[K_{t+1}]$ by discarding the second-order term $\frac{\phi}{2}\left(k_{t+2}-k_{t+1}\right)^2$ and plugging in the expression for k_{t+1} from $[I_t]$, we arrive at the system

$$[I_t] \qquad k_{t+1} = k_t + \frac{1}{\phi} q_t$$

$$[K_{t+1}] \qquad 1 + q_t = \beta \left[\frac{1-\beta}{\beta} - \kappa \left(k_t + \frac{1}{\phi} q_t \right) + 1 + q_{t+1} \right]$$

$$\iff \quad q_{t+1} = \kappa\, k_t + \left[\frac{1}{\beta} + \frac{\kappa}{\phi} \right] q_t$$

- Thus, the (log-linearized) dynamic system can be written as

$$\underbrace{\begin{pmatrix} k_{t+1} \\ q_{t+1} \end{pmatrix}}_{\equiv x_{t+1}} = \underbrace{\begin{pmatrix} 1 & \frac{1}{\phi} \\ \kappa & \frac{1}{\beta} + \frac{\kappa}{\phi} \end{pmatrix}}_{\equiv A} \underbrace{\begin{pmatrix} k_t \\ q_t \end{pmatrix}}_{\equiv x_t}.$$

Problem 3

Question 5

5. Is the Blanchard-Kahn condition for a unique solution satisfied?

Problem 3

Question 5: Solution

- **Blanchard-Kahn condition**: Let m be the $\#$ of control variables and n the number of eigenvalues outside the unit circle. Then,

Problem 3

Question 5: Solution

- **Blanchard-Kahn condition**: Let m be the $\#$ of control variables and n the number of eigenvalues outside the unit circle. Then,
  - $m = n \implies$ unique solution;

Problem 3

Question 5: Solution

- **Blanchard-Kahn condition**: Let m be the $\#$ of control variables and n the number of eigenvalues outside the unit circle. Then,
  - $m = n \implies$ unique solution;
  - $m > n \implies$ multiple solutions; and

Problem 3

Question 5: Solution

- **Blanchard-Kahn condition**: Let m be the $\#$ of control variables and n the number of eigenvalues outside the unit circle. Then,
  - $m = n \implies$ unique solution;
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  - $m < n \implies$ no solution.

Problem 3

Question 5: Solution

- **Blanchard-Kahn condition**: Let m be the # of control variables and n the number of eigenvalues outside the unit circle. Then,
  - $m = n \implies$ unique solution;
  - $m > n \implies$ multiple solutions; and
  - $m < n \implies$ no solution.
- Here, we have $m = 1$. Will need to look at the eigenvalues to find n.

Problem 3

Question 5: Solution

- **eigenvalues**: Let γ_1 and γ_2 represent the eigenvalues of matrix A (defined in previous question). An eigenvalue γ solves

$$|A - \gamma I| = 0$$

$$\iff \begin{vmatrix} 1-\gamma & \frac{1}{\phi} \\ \kappa & \frac{1}{\beta} + \frac{\kappa}{\phi} - \gamma \end{vmatrix} = 0$$

$$\iff \gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)\gamma + \frac{1}{\beta} = 0$$

Problem 3
Question 5: Solution

- Let $f(\gamma) = \gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)\gamma + \frac{1}{\beta}$ and note that

Problem 3
Question 5: Solution

- Let $f(\gamma) = \gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)\gamma + \frac{1}{\beta}$ and note that
  - $f(0) = \frac{1}{\beta} > 0$.

Problem 3
Question 5: Solution

- Let $f(\gamma) = \gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)\gamma + \frac{1}{\beta}$ and note that
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  - $f(1) = -\frac{\kappa}{\phi} < 0$.

Problem 3
Question 5: Solution

- Let $f(\gamma) = \gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)\gamma + \frac{1}{\beta}$ and note that
  - $f(0) = \frac{1}{\beta} > 0$.
  - $f(1) = -\frac{\kappa}{\phi} < 0$.
  - $f(\gamma)$ is a parabola with a min at $\frac{1}{2} + \frac{1}{2\beta} + \frac{\kappa}{2\phi} > 1$, so $(0, f(0))$ and $(1, f(1))$ lie on its decreasing side. Thus, $\gamma_1 > 1$ and $\gamma_2 \in (0, 1)$, i.e., $\boxed{n = 1}$.

Problem 3

Question 5: Solution

- The Blanchard-Kahn condition for a unique solution is satisfied since $m = n$.

Problem 3
Question 5: Solution

- The Blanchard-Kahn condition for a unique solution is satisfied since $m = n$.

- For reference,

$$\gamma_{1,2} = \frac{1}{2}\left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi} \pm \sqrt{\left(1 + \frac{1}{\beta} + \frac{\kappa}{\phi}\right)^2 - \frac{4}{\beta}}\right).$$

Problem 3
Question 6

6. Solve for the saddle path using the Blanchard-Kahn method.

Problem 3

Question 6: Solution

- If γ is an eigenvalue, its eigenvector v solves

$$(A - \gamma I)\, v = 0$$

$$\iff \begin{pmatrix} 1-\gamma & \frac{1}{\phi} \\ \kappa & \frac{1}{\beta} + \frac{\kappa}{\phi} - \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Problem 3

Question 6: Solution

- If γ is an eigenvalue, its eigenvector v solves

$$(A - \gamma I)\, v = 0$$

$$\iff \begin{pmatrix} 1-\gamma & \frac{1}{\phi} \\ \kappa & \frac{1}{\beta} + \frac{\kappa}{\phi} - \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

- Let $v_1 = 1$. Then, from the 1st equation, $v_2 = \phi\,(\gamma - 1)$. Thus, the eigenvectors are of the form

$$v = \begin{pmatrix} 1 \\ \phi\,(\gamma - 1) \end{pmatrix}.$$

Problem 3
Question 6: Solution

- The matrix of eigenvectors is then

$$Q = \begin{pmatrix} 1 & 1 \\ \phi(\gamma_1 - 1) & \phi(\gamma_2 - 1) \end{pmatrix}$$

with inverse

$$Q^{-1} = \frac{1}{\phi(\gamma_2 - \gamma_1)} \begin{pmatrix} \phi(\gamma_2 - 1) & -1 \\ -\phi(\gamma_1 - 1) & 1 \end{pmatrix}$$

Problem 3
Question 6: Solution

- Let $y_t = Q^{-1} x_t \iff x_t = Q\, y_t$. The decoupled system is then

$$\underbrace{\begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix}}_{\equiv y_{t+1}} = \underbrace{\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}}_{\equiv \Lambda} \underbrace{\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}}_{\equiv y_t}.$$

Problem 3

Question 6: Solution

- Iterating forward the 1st equation (corresponding to $\gamma_1 > 1$):

$$\begin{aligned} y_{1,t} &= \gamma_1^{-1}\, y_{1,t+1} \\ &= \gamma_1^{-2}\, y_{1,t+2} \\ &\vdots \\ &= \lim_{j\to\infty} \gamma_1^{-j}\, y_{1,t+j} \\ &= 0, \end{aligned}$$

where we assume that the TVC requires convergence to the SS, so $\lim_{j\to\infty} y_{1,t+j} = 0$.

Problem 3

Question 6: Solution

- Finally, going back to the original coordinates (i.e., $x_t = Qy_t$):

$$\begin{pmatrix} k_t \\ q_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \phi\left(\gamma_1 - 1\right) & \phi\left(\gamma_2 - 1\right) \end{pmatrix} \begin{pmatrix} 0 \\ y_{2,t} \end{pmatrix}$$

Problem 3
Question 6: Solution

- Finally, going back to the original coordinates (i.e., $x_t = Qy_t$):

$$\begin{pmatrix} k_t \\ q_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \phi\left(\gamma_1 - 1\right) & \phi\left(\gamma_2 - 1\right) \end{pmatrix} \begin{pmatrix} 0 \\ y_{2,t} \end{pmatrix}$$

- Plugging the 1st equation ($k_t = y_{2,t}$) into the second one ($q_t = \phi\left(\gamma_2 - 1\right) y_{2,t}$), we obtain

$$\boxed{q_t = -\phi\left(1 - \gamma_2\right) k_t}.$$

Problem 3
Question 7

7. Solve for the optimal dynamics of k_t.

Problem 3
Question 7: Solution

- Substituting the optimal q_t into the 1st eqn. of the original system:

$$k_{t+1} = k_t + \frac{1}{\phi} q_t = k_t - (1 - \gamma_2) k_t \iff \boxed{k_{t+1} = \gamma_2 k_t}.$$

Problem 3

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- $\gamma_2 \in (0, 1) \implies k_t \to 0$ monotonically, i.e., K_t monotonically converges to its steady state level.

Problem 3

Question 8

8. Draw a phase diagram.

Problem 3

Question 8: Solution

- Starting from the log-linearized system:

Problem 3
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Problem 3
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  - $k_{t+1} = k_t + \frac{1}{\phi} q_t \iff \boxed{\Delta k_{t+1} = \frac{1}{\phi} q_t}.$
  - $q_{t+1} = \kappa\, k_t + \left(\frac{1}{\beta} + \frac{\kappa}{\phi}\right) q_t \iff \boxed{\Delta q_{t+1} = \kappa\, k_t + \left(\frac{1}{\beta} + \frac{\kappa}{\phi} - 1\right) q_t}.$

Problem 3

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  - Thus, the iso-lines are $\boxed{q_t = 0}$ and $\boxed{q_t = \frac{\kappa}{1 - \frac{1}{\beta} - \frac{\kappa}{\phi}} k_t}$.

Problem 3

Question 8: Solution