EC400: DPDE

Class #1

Pinjas Albagli
p.m.albagli@lse.ac.uk

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Problem 1

Problem 1

Suppose that an agent gets utility not only from consumption over T periods of her life, but also from leaving a bequest to its descendants

$$\sum_{t=0}^{T} \beta^t \ln(C_t) + \eta \beta^T \ln(B_{T+1})$$

and is subject to a standard budget constraint

$$B_{t+1} = RB_t + Y_t - C_t.$$

Problem 1
Question 1

1. Is the additional constraint $B_{T+1} \geq 0$ discussed in the lecture binding in this case?

Problem 1

Question 1: Solution

- $B_{T+1} \geq 0$ is **not binding** since

$$U\left(\{C_t\}_{t=0}^{T}, B_{T+1}\right) \underset{B_{T+1}\to 0}{\longrightarrow} -\infty \implies B^*_{T+1} > 0.$$

Problem 1
Question 2

2. Derive the optimality conditions and provide intuition for the terminal condition.

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- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \ln(C_t) + \eta\beta^T \ln(B_{T+1}) + \sum_{t=0}^{T} \lambda_t (RB_t + Y_t - C_t - B_{t+1}) + \mu B_{T+1}$$

where $\mathcal{L} = \mathcal{L}\left(\{C_t, B_{t+1}, \lambda_t\}_{t=0}^{T}, \mu \mid \{Y_t\}_{t=0}^{t}, R, \beta, \eta\right)$.

Problem 1
Question 2: Solution

- FOC:

$$[C_t] \qquad \beta^t \frac{1}{C_t} - \lambda_t = 0, \qquad \forall\, t \in \{0, \dots, T\},$$

$$[B_{t+1}] \qquad -\lambda_t + R\lambda_{t+1} = 0, \qquad \forall\, t \in \{0, \dots, T-1\},$$

$$[B_{T+1}] \qquad \eta\beta^T \frac{1}{B_{T+1}} - \lambda_T = 0.$$

Problem 1

Question 2: Solution

- Solving:

Problem 1

Question 2: Solution

- Solving:
  - **Euler equations:** From $[C_t]$ and $[B_{t+1}]$, $\forall\, t \in \{0, \ldots, T-1\}\,,\ \beta^t \frac{1}{C_t} = R\beta^{t+1}\frac{1}{C_{t+1}} \iff \boxed{C_{t+1} = \beta R C_t}$.

Problem 1
Question 2: Solution

- Solving:
  - **Euler equations:** From $[C_t]$ and $[B_{t+1}]$,
    $\forall\, t \in \{0, \ldots, T-1\}\,,\ \beta^t \frac{1}{C_t} = R\beta^{t+1}\frac{1}{C_{t+1}} \iff \boxed{C_{t+1} = \beta R C_t}$.
  - **Terminal condition:** From $[C_t]$ and $[B_{T+1}]$,
    $\beta^T \frac{1}{C_T} = \eta\beta^T \frac{1}{B_{T+1}} \iff \boxed{B_{T+1} = \eta C_T}$.

Problem 1

Question 2: Solution

- Solving:
  - **Euler equations:** From $[C_t]$ and $[B_{t+1}]$,
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  - **Terminal condition:** From $[C_t]$ and $[B_{T+1}]$,
    $\beta^T \frac{1}{C_T} = \eta\beta^T \frac{1}{B_{T+1}} \iff \boxed{B_{T+1} = \eta C_T}$.
  - **Intuition for the terminal condition:** In the last period, agent equalizes marginal utility from spending money on consumption and leaving it as a bequest.

Problem 1
Question 3

3. Solve for the optimal consumption C_0 and bequest B_{T+1} assuming $B_0 = 0$ and $Y_t = Y$.

Problem 1
Question 3: Solution

- Iterating the budget constraint forward:

$$\sum_{t=0}^{T} R^{-t} \left(C_t - Y \right) + R^{-T} B_{T+1} = 0.$$

details

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Question 3: Solution

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- From the optimality conditions: $\boxed{C_t = \left(R\beta\right)^t C_0}$ for $t \in \{0, \ldots, T\}$ and $\boxed{B_{T+1} = \eta\left(R\beta\right)^T C_0}$.

details

Problem 1
Question 3: Solution

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- From the optimality conditions: $\boxed{C_t = (R\beta)^t \, C_0}$ for $t \in \{0, \ldots, T\}$ and $\boxed{B_{T+1} = \eta\,(R\beta)^T \, C_0}$.

details

- Substitute into the intertemporal BC and solve for C_0 to obtain:

$$\boxed{C_0 = \frac{1-\beta}{1+\eta\beta^T-(1+\eta)\beta^{T+1}} \frac{R-R^{-T}}{R-1} Y}.$$

details

Problem 1

Question 4

4. What happens to the optimal bequest when $T \to \infty$ if $\beta R = 1$? Explain.

Problem 1

Question 4: Solution

- Taking the limit as $T \to \infty$ on both sides of the expression for B_{T+1} and using the assumptions $\boxed{\beta < 1}$, $\boxed{R > 1}$, and $\boxed{\beta R = 1}$:

Problem 1

Question 4: Solution

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$$B_{T+1} \underset{T \to \infty}{\longrightarrow} \eta \frac{1-\beta}{1+0+0} \frac{R-0}{R-1} Y$$

Problem 1

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$$B_{T+1} \underset{T\to\infty}{\longrightarrow} \eta \frac{1-\beta}{1+0+0} \frac{R-0}{R-1} Y$$

$$= \eta \frac{R-1}{R} \frac{R}{R-1} Y$$

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$$= \eta Y.$$

Problem 1
Question 4: Solution

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Question 4: Solution

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  - Utility from the bequest is isomorphic to utility from consumption in a period far in the future.

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  - Utility from the bequest is isomorphic to utility from consumption in a period far in the future.
  - $\beta R = 1 \implies$ agent perfectly smooths consumption (i.e., $C_t = C \; \forall \; t$) and also leaves a positive bequest.

Problem 2

Problem 2

Consider an agent with a CRRA utility

$$\int_0^T e^{-\rho t} \frac{C_t^{1-\sigma} - 1}{1 - \sigma} \mathrm{d}t$$

and a budget constraint

$$\dot{B}_t = rB_t - C_t,$$

where labor income is zero, $r = \rho$, and $B_0 > 0$.

Problem 2
Question 1

1. Show that the CRRA utility converges to $u(C) = \ln(C)$ in the limit $\sigma \to 1$. Focus on this limit for the rest of the problem.

Problem 2
Question 1: Solution

- Using L'Hôspital's rule and recalling that $\frac{\mathrm{d}a^x}{\mathrm{d}x} = \ln(a) a^x$,

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Problem 2

Question 1: Solution

- Using L'Hôspital's rule and recalling that $\frac{\mathrm{d}a^x}{\mathrm{d}x} = \ln(a)a^x$,

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Problem 2

Question 1: Solution

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$$= \lim_{\sigma \to 1} \frac{e^{(1-\sigma)\ln(C)} - 1}{1-\sigma}$$

Problem 2

Question 1: Solution

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&= \lim_{\sigma \to 1} \frac{e^{(1-\sigma)\ln(C)} - 1}{1-\sigma} \\
&\overset{\text{L'H}}{=} \frac{-\ln(C)\, e^{(1-\sigma)\ln(C)}}{-1}
\end{aligned}$$

Problem 2

Question 1: Solution

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- Alternatively, using the fact that $x = e^{\ln(x)}$ and L'Hôspital's rule,

$$\begin{aligned}
\lim_{\sigma\to 1} u(C) &= \lim_{\sigma\to 1} \frac{C^{1-\sigma}-1}{1-\sigma} \\
&= \lim_{\sigma\to 1} \frac{e^{\ln(C^{1-\sigma})}-1}{1-\sigma} \\
&= \lim_{\sigma\to 1} \frac{e^{(1-\sigma)\ln(C)}-1}{1-\sigma} \\
&\overset{\text{L'H}}{=} \frac{-\ln(C)\, e^{(1-\sigma)\ln(C)}}{-1} \\
&= \ln(C).
\end{aligned}$$

Problem 2

Question 2

2. Write down the Hamiltonian and derive the optimality conditions.

Problem 2
Question 2: Solution

- Hamiltonian:

$$\mathcal{H}_t = e^{-\rho t} \ln \left(C_t \right) + \lambda_t \left(rB_t - C_t \right)$$

Problem 2

Question 2: Solution

- Hamiltonian:

$$\mathcal{H}_t = e^{-\rho t} \ln\left(C_t\right) + \lambda_t \left(rB_t - C_t\right)$$

- Maximum principle:

$$[C_t] \qquad \frac{\partial \mathcal{H}_t}{\partial C_t} = 0 \iff e^{-\rho t}\frac{1}{C_t} - \lambda_t = 0$$

$$[B_t] \qquad \frac{\partial \mathcal{H}_t}{\partial B_t} = -\dot{\lambda}_t \iff r\lambda_t = -\dot{\lambda}_t$$

Problem 2
Question 2: Solution

- Optimality conditions:

Problem 2

Question 2: Solution

- Optimality conditions:
  - From $[C_t]$: $\boxed{\lambda_t = \frac{e^{-\rho t}}{C_t}} \implies \boxed{-\dot{\lambda}_t = \frac{e^{-\rho t}}{C_t}\left[\rho + \frac{\dot{C}_t}{C_t}\right]}$.

Problem 2

Question 2: Solution

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  - Plugging into $[B_t]$:

$$r\frac{e^{-\rho t}}{C_t} = \frac{e^{-\rho t}}{C_t}\left[\rho + \frac{\dot{C}_t}{C_t}\right] \iff (r-\rho) = \frac{\dot{C}_t}{C_t} = 0 \implies \boxed{C_t = C_0 e^{(r-\rho)t}}.$$

Problem 2
Question 2: Solution

- Optimality conditions:
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  - Using the assumption $\boxed{r = \rho}$: $\boxed{C_t = C_0 \; \forall\, t \in [0, T]}$.

▸ details

Problem 2

Question 2: Solution

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  - Using the assumption $\boxed{r = \rho}$: $\boxed{C_t = C_0 \,\forall\, t \in [0,\, T]}$.

    details
  - Transversality condition:
    $\lambda_T B_T = 0 \iff \frac{e^{-\rho t}}{C_0} B_T = 0 \iff \boxed{B_T = 0}$.

Problem 2

Question 3

3. Obtain the intertemporal budget constraint and solve for the optimal consumption.

Problem 2

Question 3: Solution

- Multiplying by e^{-rt} on both sides of the flow BC $\dot{B}_t = rB_t - C_t$, integrating on both sides and evaluating at $t = T$:

Problem 2

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$$e^{-rt}\dot{B}_t - re^{-rt}B_t = -e^{-rt}C_t$$

Problem 2
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$$e^{-rt}\dot{B}_t - re^{-rt}B_t = -e^{-rt}C_t$$

$$\iff \mathrm{d}\left[e^{-rt}B_t\right] = -e^{-rt}C_t\mathrm{d}t$$

Problem 2
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$$\iff \mathrm{d}\left[e^{-rt}B_t\right] = -e^{-rt}C_t\mathrm{d}t$$

$$\iff e^{-rt}B_t = B_0 - \int_0^t e^{-rs}C_s\mathrm{d}s$$

Problem 2
Question 3: Solution

- Multiplying by e^{-rt} on both sides of the flow BC $\dot{B}_t = rB_t - C_t$, integrating on both sides and evaluating at $t = T$:

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$$\iff \mathrm{d}\left[e^{-rt}B_t\right] = -e^{-rt}C_t\mathrm{d}t$$

$$\iff e^{-rt}B_t = B_0 - \int_0^t e^{-rs}C_s\mathrm{d}s$$

$$\implies \underbrace{e^{-rT}B_T}_{0 \text{ by TVC}} = B_0 - \int_0^T e^{-rs}C_s\mathrm{d}s.$$

Problem 2
Question 3: Solution

- Using $\boxed{C_s = C_0 \,\forall\, s \in [0, T]}$ and solving for C_0:

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- Using $\boxed{C_s = C_0 \; \forall \, s \in [0, \, T]}$ and solving for C_0:

$$\int_0^T e^{-rs} C_0 \mathrm{d}s = B_0$$

Problem 2

Question 3: Solution

- Using $\boxed{C_s = C_0 \;\forall\, s \in [0,\, T]}$ and solving for C_0:

$$\int_0^T e^{-rs}\, C_0 \mathrm{d}s = B_0$$

$$\iff \quad C_0 \left[-\frac{1}{r} e^{-rs} \right] \Bigg|_{s=0}^{T} = B_0$$

Problem 2

Question 3: Solution

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- Using $\boxed{C_s = C_0 \, \forall \, s \in [0, T]}$ and solving for C_0:

$$\int_0^T e^{-rs} C_0 \mathrm{d}s = B_0$$

$$\iff C_0 \left[-\frac{1}{r} e^{-rs} \right] \Big|_{s=0}^{T} = B_0$$

$$\iff C_0 = \frac{rB_0}{1 - e^{-rT}}$$

Problem 2
Question 3: Solution

- Using $\boxed{C_s = C_0 \; \forall \, s \in [0, T]}$ and solving for C_0:

$$\int_0^T e^{-rs} C_0 \mathrm{d}s = B_0$$

$$\iff C_0 \left[-\frac{1}{r} e^{-rs} \right] \Big|_{s=0}^{T} = B_0$$

$$\iff C_0 = \frac{rB_0}{1 - e^{-rT}}$$

- Finally, using $\boxed{r = \rho} \implies \boxed{C_t = C_0 \; \forall \, t \in [0, T]}$:

$$C_t = \frac{\rho B_0}{1 - e^{-\rho T}} \; \forall t \in [0, T].$$

Problem 2

Question 4

4. Suppose the agent can choose not only consumption path, but also the length of life T. Assuming utility is zero for $t > T$ and the agent cannot leave any debt, solve for the optimal value of T.

Problem 2

Question 4: Solution

- Substituting the optimal consumption path into the objective function and integrating:

Problem 2
Question 4: Solution

- Substituting the optimal consumption path into the objective function and integrating:

$$\int_0^T e^{\rho t} \ln\left(\frac{\rho B_0}{1 - e^{-\rho T}}\right) \mathrm{d}t = \left[\ln\left(\rho B_0\right) - \ln\left(1 - e^{-\rho T}\right)\right]\left(-\frac{e^{-\rho t}}{\rho}\right)\Big|_{t=0}^{T}$$

Problem 2

Question 4: Solution

- Substituting the optimal consumption path into the objective function and integrating:

$$\int_0^T e^{\rho t} \ln\left(\frac{\rho B_0}{1-e^{-\rho T}}\right) \mathrm{d}t = \Big[\ln(\rho B_0) - \ln(1-e^{-\rho T})\Big]\Big(-\frac{e^{-\rho t}}{\rho}\Big)\Big|_{t=0}^{T}$$

$$= \Big[\ln(\rho B_0) - \ln(1-e^{-\rho T})\Big]\Big(-\frac{e^{-\rho T}+1}{\rho}\Big).$$

Problem 2

Question 4: Solution

- Substituting the optimal consumption path into the objective function and integrating:

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$$= \left[\ln(\rho B_0) - \ln\left(1-e^{-\rho T}\right)\right]\left(-\frac{e^{-\rho T}+1}{\rho}\right).$$

- The problem is:

$$\max_{T\in\mathbb{R}_+^2} \left[\ln(\rho B_0) - \ln\left(1-e^{-\rho T}\right)\right]\left(1-e^{-\rho T}\right).$$

Problem 2

Question 4: Solution

- Taking FOC and solving for T^*, we find:

$$T^* = \begin{cases} -\frac{1}{\rho}\ln\left(1-\rho B_0 e^{-1}\right) & \text{if} \quad \rho B_0 e^{-1} < 1 \\ +\infty & \text{otherwise} \end{cases},$$

where $-\rho B_0 e^{-1} < 0 \implies \ln\left(1-\rho B_0 e^{-1}\right) < 0$.

details

Problem 2

Question 5

5. Derive the optimality condition for T using a perturbation argument.

Problem 2

Question 5: Solution

- Starting from the, suppose the agent increases T by $\mathrm{d}T = \Delta > 0$.

Problem 2

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  - Given $\dot{C}_t = 0$, the agent spends $C_T \Delta$ over this period.

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- Starting from the, suppose the agent increases T by $\mathrm{d}T = \Delta > 0$.
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  - This requires a proportionate increase in B_T.

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- Starting from the, suppose the agent increases T by $\mathrm{d}T = \Delta > 0$.
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  - According to the BC:

Problem 2

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  - Given $\dot{C}_t = 0$, the agent spends $C_T \Delta$ over this period.
  - This requires a proportionate increase in B_T.
  - According to the BC:

$$B_T = e^{rT} B_0 - \frac{1}{r}\left(1 - e^{-rT}\right) C_T$$

Problem 2

Question 5: Solution

- Starting from the, suppose the agent increases T by $\mathrm{d}T = \Delta > 0$.
  - Given $\dot{C}_t = 0$, the agent spends $C_T \Delta$ over this period.
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  - According to the BC:

$$B_T = e^{rT} B_0 - \frac{1}{r}\left(1 - e^{-rT}\right) C_T$$

$$\implies \underbrace{\mathrm{d}B_T}_{C_T\Delta} = e^{rT} B_0 - \frac{1}{r}\left(1 - e^{-rT}\right) \mathrm{d}C_T$$

Problem 2

Question 5: Solution

- Starting from the, suppose the agent increases T by $\mathrm{d}T = \Delta > 0$.
  - Given $\dot{C}_t = 0$, the agent spends $C_T\Delta$ over this period.
  - This requires a proportionate increase in B_T.
  - According to the BC:

$$B_T = e^{rT} B_0 - \frac{1}{r}\left(1 - e^{-rT}\right) C_T$$

$$\Longrightarrow \underbrace{\mathrm{d}B_T}_{C_T\Delta} = e^{rT} B_0 - \frac{1}{r}\left(1 - e^{-rT}\right) \mathrm{d}C_T$$

$$\Longleftrightarrow \mathrm{d}C_T = -\frac{rC_T\Delta}{e^{rT} - 1}.$$

Problem 2

Question 5: Solution

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Question 5: Solution

- Thus,

$$\boxed{\mathrm{d}C_t = -\frac{rC_T\Delta}{e^{rT}-1} \; \forall t \in [0,\, T]}$$

- Thus,

$$\boxed{\mathrm{d}C_t = -\frac{rC_T\Delta}{e^{rT}-1}\ \forall t \in [0,\, T]}$$

- Given $r = \rho$, the net change in utility is

$$e^{-\rho T}u(C_T)\Delta + \int_0^T e^{-\rho t}u'(C_T)\mathrm{d}C_T\mathrm{d}t = e^{-\rho T}\Big[\ln\left(C_T\right) - 1\Big]\Delta.$$

details

Problem 2

Question 5: Solution

- Under the interior solution, we must have

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$$\mathrm{d}u = 0$$

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- Under the interior solution, we must have

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- Finally, combining with our previous expression for C_T,

Problem 2
Question 5: Solution

- Under the interior solution, we must have

$$\mathrm{d}u = 0 \iff \ln\left(C_T\right) \quad = 1 \iff C_T = e.$$

- Finally, combining with our previous expression for C_T,

$$\frac{\rho B_0}{1 - e^{-\rho T}} = e$$

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- Under the interior solution, we must have

$$\mathrm{d}u = 0 \iff \ln\left(C_T\right) \quad = 1 \iff C_T = e.$$

- Finally, combining with our previous expression for C_T,

$$\frac{\rho B_0}{1 - e^{-\rho T}} = e$$

$$\iff e^{-\rho T} = 1 - \rho B_0 e^{-1}$$

Problem 2

Question 5: Solution

- Under the interior solution, we must have

$$\mathrm{d}u = 0 \iff \ln\left(C_T\right) = 1 \iff C_T = e.$$

- Finally, combining with our previous expression for C_T,

$$\frac{\rho B_0}{1 - e^{-\rho T}} = e$$

$$\iff e^{-\rho T} = 1 - \rho B_0 e^{-1}$$

$$\iff T = -\frac{1}{\rho} \ln\left(1 - \rho B_0 e^{-1}\right).$$

Problem 3

Problem 3

Consider the following investment problem. Given initial value of capital K_0, a firm chooses investment path $\{I_t\}_{t=0}^{\infty}$ to maximize profits

$$\sum_{t=0}^{\infty} \beta^t \left[A_t K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right]$$

subject to the capital law of motion

$$K_{t+1} = (1 - \delta) K_t + I_t,$$

where A_t is the firm's productivity, $\alpha \in (0, 1)$, $\phi > 0$ stays for the capital adjustment costs, and $\delta \in (0, 1)$ is the depreciation rate.

Problem 3
Question 1

1. Write down the Lagrangian and take the first-order conditions.

Problem 3
Question 1: Solution

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[A_t K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right] + \sum_{t=0}^{\infty} \lambda_t \left[(1-\delta) K_t + I_t - K_{t+1} \right]$$

Problem 3

Question 1: Solution

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[A_t K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right] + \sum_{t=0}^{\infty} \lambda_t \left[(1-\delta) K_t + I_t - K_{t+1} \right]$$

- FOC:

$$[I_t] \qquad -\beta^t \left(1 + \phi \frac{I_t}{K_t} \right) + \lambda_t = 0$$

$$[K_{t+1}] \qquad -\lambda_t + \beta^{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + \frac{\phi}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) + \lambda_{t+1}(1-\delta) = 0$$

Problem 3
Question 2

2. Use a finite-period version of the model to derive the transversality condition.

Problem 3
Question 2: Solution

- In a finite-period version of the model, we add that the firm **cannot leave a negative stock of capital**, i.e., $\boxed{K_{T+1} \geq 0}$.

Problem 3

Question 2: Solution

- In a finite-period version of the model, we add that the firm **cannot leave a negative stock of capital**, i.e., $\boxed{K_{T+1} \geq 0}$.

- The Lagrangian of this problem is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[A_t K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right] + \sum_{t=0}^{T} \lambda_t \left[(1-\delta) K_t + I_t - K_{t+1} \right] + \mu K_{T+1}$$

Problem 3
Question 2: Solution

- In a finite-period version of the model, we add that the firm **cannot leave a negative stock of capital**, i.e., $\boxed{K_{T+1} \geq 0}$.
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$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[A_t K_t^{\alpha} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \right] + \sum_{t=0}^{T} \lambda_t \left[(1-\delta) K_t + I_t - K_{t+1} \right] + \mu K_{T+1}$$

- The FOC wrt K_{T+1} is $-\lambda_T + \mu = 0 \iff \boxed{\mu = \lambda_T}$.

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- The complementary slackness condition can then be written as $\mu K_{t+1} = 0 \iff \boxed{\lambda_T K_{T+1} = 0}$.

Problem 3

Question 2: Solution

- The complementary slackness condition can then be written as $\mu K_{t+1} = 0 \iff \boxed{\lambda_T K_{T+1} = 0}$.

- By analogy, the **transversality condition** in the infinite-horizon model is $\boxed{\lim_{t \to \infty} \lambda_t K_{t+1} = 0}$.

Problem 3

Question 3

3. Derive the analog of the Euler equation, i.e. the optimality condition for capital and investment that does not include λ_t. Interpret this equation using the perturbations of the optimal path.

Problem 3

Question 3: Solution

- Combining $[I_t]$ and $[K_{t+1}]$:

Problem 3
Question 3: Solution

- Combining $[I_t]$ and $[K_{t+1}]$:

$$\beta^t\left(1+\phi\frac{I_t}{K_t}\right)=\beta^{t+1}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}+\frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\right)+(1-\delta)\beta^{t+1}\left(1+\phi\frac{I_{t+1}}{K_{t+1}}\right)$$

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$$\beta^t\left(1+\phi\frac{I_t}{K_t}\right)=\beta^{t+1}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}+\frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\right)+(1-\delta)\beta^{t+1}\left(1+\phi\frac{I_{t+1}}{K_{t+1}}\right)$$

$$\iff 1+\phi\frac{I_t}{K_t}=\beta\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}+\frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\right)+(1-\delta)\beta\left(1+\phi\frac{I_{t+1}}{K_{t+1}}\right).$$

- **Perturbation argument**: Suppose the firm increases I_t by $\mathrm{d}I_t = \Delta > 0$, returning to the optimal trajectory in $t + 2$.

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  - $\mathrm{d}I_t = \Delta \implies \mathrm{d}\psi_t = \left(1 + \phi\frac{I_t}{K_t}\right)\mathrm{d}I_t = \left(1 + \phi\frac{I_t}{K_t}\right)\Delta > 0$, where $\psi_t = I_t + \frac{\phi}{2}\frac{I_t^2}{K_t}$.

Problem 3
Question 3: Solution

- **Perturbation argument**: Suppose the firm increases I_t by $\mathrm{d}I_t = \Delta > 0$, returning to the optimal trajectory in $t+2$.
  - $\mathrm{d}I_t = \Delta \implies \mathrm{d}\psi_t = \left(1 + \phi \frac{I_t}{K_t}\right) \mathrm{d}I_t = \left(1 + \phi \frac{I_t}{K_t}\right) \Delta > 0$, where $\psi_t = I_t + \frac{\phi}{2} \frac{I_t^2}{K_t}$.
  - $\mathrm{d}I_t = \Delta \implies \mathrm{d}K_{t+1} = \Delta$. Thus, $\mathrm{d}\pi_{t+1} = \frac{\partial \pi_{t+1}}{\partial K_{t+1}} \mathrm{d}K_{t+1} = \alpha A_{t+1} K_{t+1}^{\alpha-1} \Delta > 0$, where $\pi_t = A_t K_t^{\alpha}$.

- Returning to the optimal trajectory in $t+2$ means leaving K_{t+2} uncahanged, i.e., $\mathrm{d}K_{t+2} = 0$. Therefore,

$$\underbrace{\mathrm{d}K_{t+2}}_{0} = (1-\delta)\underbrace{\mathrm{d}K_{t+1}}_{\Delta} + \mathrm{d}I_{t+1} \iff \mathrm{d}I_{t+1} = -(1-\delta)\Delta < 0.$$

Problem 3
Question 3: Solution

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- $\mathrm{d}K_{t+1}=\Delta$ and $\mathrm{d}I_{t+1}=-(1-\delta)\,\Delta \implies$
$\mathrm{d}\psi_{t+1}=-(1-\delta)\left(1+\phi\frac{I_{t+1}}{K_{t+1}}\right)\Delta-\frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\Delta<0$

Problem 3

Question 3: Solution

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- $\mathrm{d}K_{t+1} = \Delta$ and $\mathrm{d}I_{t+1} = -(1-\delta)\,\Delta \implies$
$\mathrm{d}\psi_{t+1} = -(1-\delta)\left(1+\phi\frac{I_{t+1}}{K_{t+1}}\right)\Delta - \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2}\Delta < 0$

- Combining all these costs and benefits and requiring that the perturbation is not profitable, we get the Euler equation.

Problem 3
Question 4

4. Rewrite the capital law of motion for an arbitrary length of period Δ. Obtain the continuous-time version of the optimization problem.

Problem 3
Question 4

- Note that K_{t+1} is a **stock** variable while I_t is a **flow**, just as depreciation δK_t and profits $A_t K_t^\alpha$.

Problem 3

Question 4

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- Let $\beta \equiv e^{-r}$. Then,

Problem 3

Question 4

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- Let $\beta \equiv e^{-r}$. Then,

$$K_{t+\Delta} = (1 - \delta\Delta)K_t + I_t\Delta$$

Problem 3
Question 4

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- Let $\beta \equiv e^{-r}$. Then,

$$K_{t+\Delta} = (1 - \delta\Delta)K_t + I_t\Delta$$

$$\iff K_{t+\Delta} - K_t = (I_t - \delta K_t)\Delta$$

Problem 3
Question 4

- Note that K_{t+1} is a **stock** variable while I_t is a **flow**, just as depreciation δK_t and profits $A_t K_t^{\alpha}$.
- Let $\beta \equiv e^{-r}$. Then,

$$K_{t+\Delta} = (1-\delta\Delta)K_t + I_t\Delta$$

$$\iff K_{t+\Delta} - K_t = (I_t - \delta K_t)\Delta$$

$$\implies \dot{K}_t \equiv \lim_{\Delta\to 0} \frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t.$$

Problem 3
Question 4

- The Objective function is

$$\int_0^{\infty} e^{-rt}\left(A_t K_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t}\right)\mathrm{d}t.$$

Problem 3
Question 5

5. Define the Hamiltonian and write down the optimality conditions.

Problem 3
Question 5

- Hamiltonian:
$$\mathcal{H}_t = e^{-rt}\left(A_t K_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t}\right) + \lambda_t(I_t - \delta K_t),$$
where λ_t is the co-state variable.

Problem 3
Question 5

- Hamiltonian:

$$\mathcal{H}_t = e^{-rt}\left(A_t K_t^{\alpha} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t}\right) + \lambda_t(I_t - \delta K_t),$$

where λ_t is the co-state variable.

- Optimality conditions:

$$[I_t] \qquad \frac{\partial \mathcal{H}_t}{\partial I_t} = 0 \iff -e^{-rt}\left(1 + \phi\frac{I_t}{K_t}\right) + \lambda_t = 0$$

$$[K_t] \qquad \frac{\partial \mathcal{H}_t}{\partial K_t} = -\dot{\lambda}_t \iff e^{-rt}\left(\alpha A_t K_t^{\alpha-1} + \phi\frac{I_t^2}{K_t^2}\right) - \delta\lambda_t = -\dot{\lambda}_t$$

Problem 3
Question 6

6. Show that the same continuous-time optimality conditions can be obtained directly from the optimality conditions in discrete time.

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Problem 3

- Rewriting the FOC of the discrete-time problem for am arbitrary length of period Δ,

$[I_t]$

$$-e^{-rt}\left(1+\phi\frac{I_t}{K_t}\right)\Delta+\lambda_t\Delta=0$$

$$\iff \quad -e^{-rt}\left(1+\phi\frac{I_t}{K_t}\right)+\lambda_t=0$$

$[K_{t+1}]$

$$-\lambda_t+e^{-r(t+\Delta)}\left(\alpha A_{t+\Delta}K_{t+\Delta}^{\alpha-1}+\frac{\phi}{2}\frac{I_{t+\Delta}^2}{K_{t+\Delta}^2}\right)\Delta+\lambda_{t+\Delta}(1-\delta\Delta)=0$$

$$\iff \quad e^{-r(t+\Delta)}\left(\alpha A_{t+\Delta}K_{t+\Delta}^{\alpha-1}+\frac{\phi}{2}\frac{I_{t+\Delta}^2}{K_{t+\Delta}^2}\right)-\delta\lambda_{t+\Delta}=-\frac{\lambda_{t+\Delta}-\lambda_t}{\Delta}$$

Problem 3
Question 6

- Finally, taking limits as $\Delta \to 0$, we obtain the same optimality conditions from question 5.

Appendix

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}(B_{t+1} - Y_t + C_t)$, evaluate at $t = 0$, and iterate forward:

$$B_0 = R^{-1}(B_1 - Y_0 + C_0)$$

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}(B_{t+1} - Y_t + C_t)$, evaluate at $t = 0$, and iterate forward:

$$B_0 = R^{-1}(B_1 - Y_0 + C_0)$$

$$= R^{-1}\left[R^{-1}(B_2 - Y_1 + C_1) - Y_0 + C_0\right]$$

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}(B_{t+1} - Y_t + C_t)$, evaluate at $t = 0$, and iterate forward:

$$\begin{aligned} B_0 &= R^{-1}(B_1 - Y_0 + C_0) \\ &= R^{-1}\left[R^{-1}(B_2 - Y_1 + C_1) - Y_0 + C_0\right] \\ &= R^{-2}B_2 + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0) \end{aligned}$$

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}(B_{t+1} - Y_t + C_t)$, evaluate at $t = 0$, and iterate forward:

$$\begin{aligned}
B_0 &= R^{-1}(B_1 - Y_0 + C_0) \\
&= R^{-1}\left[R^{-1}(B_2 - Y_1 + C_1) - Y_0 + C_0\right] \\
&= R^{-2}B_2 + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0) \\
&= R^{-2}\left[R^{-1}(B_3 - Y_2 + C_2)\right] + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0)
\end{aligned}$$

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}\left(B_{t+1} - Y_t + C_t\right)$, evaluate at $t = 0$, and iterate forward:

$$\begin{aligned}
B_0 &= R^{-1}\left(B_1 - Y_0 + C_0\right) \\
&= R^{-1}\left[R^{-1}\left(B_2 - Y_1 + C_1\right) - Y_0 + C_0\right] \\
&= R^{-2} B_2 + R^{-2}\left(C_1 - Y_1\right) + R^{-1}\left(C_0 - Y_0\right) \\
&= R^{-2}\left[R^{-1}\left(B_3 - Y_2 + C_2\right)\right] + R^{-2}\left(C_1 - Y_1\right) + R^{-1}\left(C_0 - Y_0\right) \\
&= R^{-3} B_3 + R^{-3}\left(C_2 - Y_2\right) + R^{-2}\left(C_1 - Y_1\right) + R^{-1}\left(C_0 - Y_0\right)
\end{aligned}$$

Problem 1, Question 3: Details

Write the BC for $t \in \{0, \ldots, T\}$ as $B_t = R^{-1}(B_{t+1} - Y_t + C_t)$, evaluate at $t = 0$, and iterate forward:

$$\begin{aligned}
B_0 &= R^{-1}(B_1 - Y_0 + C_0) \\
&= R^{-1}\left[R^{-1}(B_2 - Y_1 + C_1) - Y_0 + C_0\right] \\
&= R^{-2}B_2 + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0) \\
&= R^{-2}\left[R^{-1}(B_3 - Y_2 + C_2)\right] + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0) \\
&= R^{-3}B_3 + R^{-3}(C_2 - Y_2) + R^{-2}(C_1 - Y_1) + R^{-1}(C_0 - Y_0) \\
&\vdots \\
&= R^{-1}\left[R^{-T}B_{T+1} + \sum_{t=0}^{T} R^{-t}(C_t - Y_t)\right].
\end{aligned}$$

Problem 1, Question 3: Details

- Thus,

$$RB_0 = R^{-T} B_{T+1} + \sum_{t=0}^{T} R^{-t} \left(C_t - Y_t \right).$$

Problem 1, Question 3: Details

- Thus,

$$RB_0 = R^{-T} B_{T+1} + \sum_{t=0}^{T} R^{-t} (C_t - Y_t) .$$

- Finally, use the assumptions $\boxed{B_0 = 0}$ and $\boxed{Y_t = Y \ \forall \, t \in \{0, \ldots, T\}}$ to obtain

$$0 = R^{-T} B_{T+1} + \sum_{t=0}^{T} R^{-t} (C_t - Y) .$$

return

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$C_t = R\beta C_{t-1}$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$
\begin{aligned}
C_t &= R\beta C_{t-1} \\
&= R\beta \left(R\beta C_{t-2} \right)
\end{aligned}
$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$\begin{aligned} C_t &= R\beta C_{t-1} \\ &= R\beta \left(R\beta C_{t-2} \right) \\ &= \left(R\beta \right)^2 C_{t-2} \end{aligned}$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$\begin{aligned} C_t &= R\beta C_{t-1} \\ &= R\beta \left(R\beta C_{t-2}\right) \\ &= \left(R\beta\right)^2 C_{t-2} \\ &= \left(R\beta\right)^2 \left(R\beta C_{t-3}\right) \end{aligned}$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$
\begin{aligned}
C_t &= R\beta C_{t-1} \\
&= R\beta \left(R\beta C_{t-2}\right) \\
&= (R\beta)^2 \, C_{t-2} \\
&= (R\beta)^2 \left(R\beta C_{t-3}\right) \\
&= (R\beta)^3 \, C_{t-3}
\end{aligned}
$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$
\begin{aligned}
C_t &= R\beta C_{t-1} \\
&= R\beta \left(R\beta C_{t-2} \right) \\
&= \left(R\beta \right)^2 C_{t-2} \\
&= \left(R\beta \right)^2 \left(R\beta C_{t-3} \right) \\
&= \left(R\beta \right)^3 C_{t-3} \\
&\vdots \\
&= \left(R\beta \right)^t C_{t-t}.
\end{aligned}
$$

Problem 1, Question 3: Details

- Just iterate backward the Euler equation:

$$\begin{aligned} C_t &= R\beta C_{t-1} \\ &= R\beta \left(R\beta C_{t-2}\right) \\ &= \left(R\beta\right)^2 C_{t-2} \\ &= \left(R\beta\right)^2 \left(R\beta C_{t-3}\right) \\ &= \left(R\beta\right)^3 C_{t-3} \\ &\vdots \\ &= \left(R\beta\right)^t C_{t-t}. \end{aligned}$$

- Finally, plug the resulting expression for C_T $\left(C_0\right)$ into the terminal condition to obtain $B_{T+1} = \eta \left(R\beta\right)^T C_0$.

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t\left(C_0\right)$ and $B_{T+1}\left(C_0\right)$ into the budget constraint:

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t(C_0)$ and $B_{T+1}(C_0)$ into the budget constraint:

$$R^{-1}\left[R^{-T}\eta\,(R\beta)^{T}\,C_0+\sum_{t=0}^{T}R^{-t}\left[(R\beta)^{t}\,C_0-Y\right]\right]=0$$

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t\left(C_0\right)$ and $B_{T+1}\left(C_0\right)$ into the budget constraint:

$$R^{-1}\left[R^{-T}\eta\left(R\beta\right)^{T}C_0+\sum_{t=0}^{T}R^{-t}\left[\left(R\beta\right)^{t}C_0-Y\right]\right]=0$$

$$\iff \eta\beta^{T}C_0+C_0\sum_{t=0}^{T}\beta^{t}-Y\sum_{t=0}^{T}R^{-t}=0$$

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t\left(C_0\right)$ and $B_{T+1}\left(C_0\right)$ into the budget constraint:

$$R^{-1}\left[R^{-T}\eta\left(R\beta\right)^{T}C_0+\sum_{t=0}^{T}R^{-t}\left[\left(R\beta\right)^{t}C_0-Y\right]\right]=0$$

$$\iff \eta\beta^{T}C_0+C_0\sum_{t=0}^{T}\beta^{t}-Y\sum_{t=0}^{T}R^{-t}=0$$

$$\iff \eta\beta^{T}C_0+C_0\frac{1-\beta^{T+1}}{1-\beta}-Y\frac{1-R^{-(T+1)}}{1-R^{-1}}=0$$

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t\left(C_0\right)$ and $B_{T+1}\left(C_0\right)$ into the budget constraint:

$$R^{-1}\left[R^{-T}\eta\left(R\beta\right)^{T}C_0+\sum_{t=0}^{T}R^{-t}\left[\left(R\beta\right)^{t}C_0-Y\right]\right]=0$$

$$\iff \eta\beta^{T}C_0+C_0\sum_{t=0}^{T}\beta^{t}-Y\sum_{t=0}^{T}R^{-t}=0$$

$$\iff \eta\beta^{T}C_0+C_0\frac{1-\beta^{T+1}}{1-\beta}-Y\frac{1-R^{-(T+1)}}{1-R^{-1}}=0$$

$$\iff C_0\frac{1-\beta^{T+1}+(1-\beta)\eta\beta^{T}}{1-\beta}=Y\frac{R\left(1-R^{-(T+1)}\right)}{R-1}$$

return

Problem 1, Question 3: Details

- Plugging in the expressions for $C_t(C_0)$ and $B_{T+1}(C_0)$ into the budget constraint:

$$R^{-1}\left[R^{-T}\eta\,(R\beta)^{T}\,C_0+\sum_{t=0}^{T}R^{-t}\left[(R\beta)^{t}\,C_0-Y\right]\right]=0$$

$$\iff \eta\beta^{T}C_0+C_0\sum_{t=0}^{T}\beta^{t}-Y\sum_{t=0}^{T}R^{-t}=0$$

$$\iff \eta\beta^{T}C_0+C_0\frac{1-\beta^{T+1}}{1-\beta}-Y\frac{1-R^{-(T+1)}}{1-R^{-1}}=0$$

$$\iff C_0\frac{1-\beta^{T+1}+(1-\beta)\eta\beta^{T}}{1-\beta}=Y\frac{R\left(1-R^{-(T+1)}\right)}{R-1}$$

$$\iff C_0=\frac{1-\beta}{1+\eta\beta^{T}-(1+\eta)\beta^{T+1}}\,\frac{R-R^{-T}}{R-1}\,Y.$$

return

Problem 2, Question 2: Details

- Rewrite the differential equation

$$\frac{\dot{C}_t}{C_t} = (r-\rho) \iff \frac{1}{C_t}\,\mathrm{d}C_t = (r-\rho)\,\mathrm{d}t$$

and integrate on both sides to obtain

$$\kappa + \ln(C_t) = \psi + (r-\rho)t$$

$$\iff \; C_t = C_0\, e^{(r-\rho)t},$$

where κ and ψ are the constants of integration and $C_0 \equiv e^{\psi-\kappa}$.

return

Problem 2, Question 4: Details

- FOC:

return

Problem 2, Question 4: Details

- FOC:

$$\frac{-\rho e^{-\rho T^*}}{1-e^{-\rho T^*}}\left[1-e^{-\rho T^*}\right]+\left[\ln\left(\rho B_0\right)-\ln\left(1-e^{-\rho T^*}\right)\right]\rho e^{-\rho T^*}=0$$

return

Problem 2, Question 4: Details

- FOC:

$$\frac{-\rho e^{-\rho T^*}}{1 - e^{-\rho T^*}} \left[1 - e^{-\rho T^*}\right] + \left[\ln(\rho B_0) - \ln(1 - e^{-\rho T^*})\right] \rho e^{-\rho T^*} = 0$$

$$\iff \ln(1 - e^{-\rho T^*}) = \ln(\rho B_0 e^{-1})$$

return

Problem 2, Question 4: Details

- FOC:

$$\frac{-\rho e^{-\rho T^*}}{1 - e^{-\rho T^*}} \left[1 - e^{-\rho T^*}\right] + \left[\ln\left(\rho B_0\right) - \ln\left(1 - e^{-\rho T^*}\right)\right] \rho e^{-\rho T^*} = 0$$

$$\iff \ln\left(1 - e^{-\rho T^*}\right) = \ln\left(\rho B_0 e^{-1}\right)$$

$$\iff e^{-\rho T^*} = 1 - \rho B_0 e^{-1}$$

return

Problem 2, Question 4: Details

- FOC:

$$\frac{-\rho e^{-\rho T^*}}{1 - e^{-\rho T^*}} \left[1 - e^{-\rho T^*}\right] + \left[\ln(\rho B_0) - \ln(1 - e^{-\rho T^*})\right] \rho e^{-\rho T^*} = 0$$

$$\iff \ln(1 - e^{-\rho T^*}) = \ln(\rho B_0 e^{-1})$$

$$\iff e^{-\rho T^*} = 1 - \rho B_0 e^{-1}$$

$$\iff T^* = -\frac{1}{\rho} \ln(1 - \rho B_0 e^{-1}),$$

return

Problem 2, Question 4: Details

- FOC:

$$\frac{-\rho e^{-\rho T^*}}{1-e^{-\rho T^*}}\left[1-e^{-\rho T^*}\right]+\left[\ln(\rho B_0)-\ln\left(1-e^{-\rho T^*}\right)\right]\rho e^{-\rho T^*}=0$$

$$\iff \ln\left(1-e^{-\rho T^*}\right)=\ln\left(\rho B_0 e^{-1}\right)$$

$$\iff e^{-\rho T^*}=1-\rho B_0 e^{-1}$$

$$\iff T^*=-\frac{1}{\rho}\ln\left(1-\rho B_0 e^{-1}\right),$$

which is well-defined as long as $1-\rho B_0 e^{-1}>0 \iff \rho B_0 e^{-1}<1$.

return

Problem 2, Question 5: Details

$$\mathrm{d}u = e^{-\rho T} \ln\left(C_T\right)\Delta - \int_0^T \frac{e^{-\rho t}}{C_T} \frac{r\, C_T \Delta}{e^{rt} - 1} \mathrm{d}t$$

return

Problem 2, Question 5: Details

$$\mathrm{d}u = e^{-\rho T} \ln(C_T)\Delta - \int_0^T \frac{e^{-\rho t}}{C_T} \frac{r C_T \Delta}{e^{rt} - 1} \mathrm{d}t$$

$$= e^{-\rho T} \ln(C_T)\Delta - \frac{r\Delta}{e^{rT} - 1} \frac{1 - e^{-\rho T}}{\rho}$$

return

Problem 2, Question 5: Details

$$
\begin{aligned}
\mathrm{d}u &= e^{-\rho T} \ln\left(C_T\right)\Delta - \int_0^T \frac{e^{-\rho t}}{C_T} \frac{r C_T \Delta}{e^{rt}-1} \mathrm{d}t \\
&= e^{-\rho T} \ln\left(C_T\right)\Delta - \frac{r\Delta}{e^{rT}-1} \frac{1-e^{-\rho T}}{\rho} \\
&= e^{-\rho T} \ln\left(C_T\right)\Delta - \frac{1-e^{-\rho T}}{\frac{1}{e^{-rT}}-1}\Delta
\end{aligned}
$$

return

Problem 2, Question 5: Details

$$
\begin{aligned}
\mathrm{d}u &= e^{-\rho T}\ln\left(C_T\right)\Delta - \int_0^T \frac{e^{-\rho t}}{C_T}\frac{r\,C_T\Delta}{e^{rt}-1}\mathrm{d}t \\
&= e^{-\rho T}\ln\left(C_T\right)\Delta - \frac{r\Delta}{e^{rT}-1}\frac{1-e^{-\rho T}}{\rho} \\
&= e^{-\rho T}\ln\left(C_T\right)\Delta - \frac{1-e^{-\rho T}}{\frac{1}{e^{-rT}}-1}\Delta \\
&= e^{-\rho T}\ln\left(C_T\right)\Delta - \frac{e^{-rT}\left(1-e^{-\rho T}\right)}{1-e^{-rT}}\Delta
\end{aligned}
$$

return

Problem 2, Question 5: Details

$$
\begin{aligned}
\mathrm{d}u &= e^{-\rho T}\ln(C_T)\Delta - \int_0^T \frac{e^{-\rho t}}{C_T}\frac{r C_T \Delta}{e^{rt}-1}\mathrm{d}t \\
&= e^{-\rho T}\ln(C_T)\Delta - \frac{r\Delta}{e^{rT}-1}\frac{1-e^{-\rho T}}{\rho} \\
&= e^{-\rho T}\ln(C_T)\Delta - \frac{1-e^{-\rho T}}{\frac{1}{e^{-rT}}-1}\Delta \\
&= e^{-\rho T}\ln(C_T)\Delta - \frac{e^{-rT}\left(1-e^{-\rho T}\right)}{1-e^{-rT}}\Delta \\
&= e^{-\rho T}\Big[\ln(C_T)-1\Big]\Delta.
\end{aligned}
$$

return