Some Useful Results

EC309 – Lent term

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Contents

1 Density of continuous monotonic transformations of continuous r.v.'s

Let $X \in \mathbb{R}$ be a continuous random variable with cdf $F_X(x) \equiv \mathbb{P}(X \leq x)$ and pdf $f_X(x) \equiv \frac{\mathrm{d}\, F_X(x)}{\mathrm{d}\, x}$, and $Y = g(X)$ with with either $g'(x) \equiv \frac{\mathrm{d}\, g(x)}{\mathrm{d}\, x} > 0\ \forall x \in \operatorname{Supp}(X)$ or $g'(x) < 0\ \forall x \in \operatorname{Supp}(X)$. We can derive the cdf and pdf of Y from those of X by noticing that

$$
\begin{aligned}
F_Y(y) &\equiv \mathbb{P}(Y \leq y) && \text{(definition)} \\
&= \mathbb{P}(g(X) \leq y) && (Y = g(X)) \\
&= \begin{cases} \mathbb{P}\left(X \leq g^{-1}(y)\right) & \text{if } g'(x) > 0\ \forall x \in \operatorname{Supp}(X) \\ \mathbb{P}\left(X \geq g^{-1}(y)\right) & \text{if } g'(x) < 0\ \forall x \in \operatorname{Supp}(X) \end{cases} && \text{(monotonicity)} \\
&= \begin{cases} \mathbb{P}\left(X \leq g^{-1}(y)\right) & \text{if } g'(x) > 0\ \forall x \in \operatorname{Supp}(X) \\ 1 - \mathbb{P}\left(X < g^{-1}(y)\right) & \text{if } g'(x) < 0\ \forall x \in \operatorname{Supp}(X) \end{cases} && \text{(event equivalence)} \\
&= \begin{cases} F_X\left(g^{-1}(y)\right) & \text{if } g'(x) > 0\ \forall x \in \operatorname{Supp}(X) \\ 1 - F_X\left(g^{-1}(y)\right) & \text{if } g'(x) < 0\ \forall x \in \operatorname{Supp}(X) \end{cases} && \text{(definition)}
\end{aligned}
$$

$$
\begin{aligned}
\implies f_Y(y) &\equiv \frac{\mathrm{d}\, F_Y(y)}{\mathrm{d}\, y} && \text{(definition)} \\
&= \begin{cases} \frac{\mathrm{d}\, F_X\left(g^{-1}(y)\right)}{\mathrm{d}\, y} & \text{if } g'(x) > 0\ \forall x \in \operatorname{Supp}(X) \\ \frac{\mathrm{d}\left(1 - F_X\left(g^{-1}(y)\right)\right)}{\mathrm{d}\, y} & \text{if } g'(x) < 0\ \forall x \in \operatorname{Supp}(X) \end{cases} && \text{(above result)} \\
&= \begin{cases} f_X\left(g^{-1}(y)\right) \frac{\mathrm{d}\, g^{-1}(y)}{\mathrm{d}\, y} & \text{if } g'(x) > 0\ \forall x \in \operatorname{Supp}(X) \\ -f_X\left(g^{-1}(y)\right) \frac{\mathrm{d}\, g^{-1}(y)}{\mathrm{d}\, y} & \text{if } g'(x) < 0\ \forall x \in \operatorname{Supp}(X) \end{cases} && \text{(chain rule)} \\
&= f_X\left(g^{-1}(y)\right) \left| \frac{\mathrm{d}\, g^{-1}(y)}{\mathrm{d}\, y} \right|
\end{aligned}
$$

2 Representation of $\mathrm{N}(\mu, \sigma^2)$ in terms of $\mathrm{N}(0,1)$ distribution

Let $\Phi(x)$ and $\phi(x)$ represent the cdf and pdf of the $\mathrm{N}(0,1)$ distribution, respectively. That is,

$$\phi(x) = (2\pi)^{-1/2} \exp\left(-\frac{x^2}{2}\right)$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(s)\,\mathrm{d}s.$$

Consider random variable $Y \sim \mathrm{N}\left(\mu, \sigma^2\right)$ and notice that

$$f_Y(y) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \qquad (Y \sim \mathrm{N}(\mu, \sigma^2))$$

$$= \sigma^{-1}\,(2\pi)^{-1/2} \exp\left(-\frac{z^2}{2}\right) \qquad (\text{c.o.v. } z = (y-\mu)/\sigma)$$

$$= \sigma^{-1}\phi(z) \qquad (\text{definition})$$

$$= \sigma^{-1}\phi\left(\frac{y-\mu}{\sigma}\right) \qquad (\text{reverse c.o.v.})$$

Also, using this result,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(s)\,\mathrm{d}s \qquad (\text{definition})$$

$$= \int_{-\infty}^{y} \sigma^{-1}\phi\left(\frac{s-\mu}{\sigma}\right)\mathrm{d}s \qquad (\text{above result})$$

$$= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \sigma^{-1}\phi\left(z\right)\left(\sigma\,\mathrm{d}z\right) \qquad (\text{c.o.v. } z = (s-\mu)/\sigma)$$

$$= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \phi\left(z\right)\mathrm{d}z$$

$$= \Phi\left(\frac{y-\mu}{\sigma}\right) \qquad (\text{definition})$$

Finally, a remarkable result:

$$\frac{\mathrm{d}\,\phi(x)}{\mathrm{d}\,x} = \frac{\mathrm{d}\,\left((2\pi)^{-1/2}\exp\left(-\frac{x^2}{2}\right)\right)}{\mathrm{d}\,x} \qquad \text{(definition)}$$

$$= -x\ (2\pi)^{-1/2}\exp\left(-\frac{x^2}{2}\right) \qquad \text{(chain rule)}$$

$$= -x\,\phi(x) \qquad \text{(definition)}$$

Moreover,

$$\frac{\mathrm{d}^2\,\phi(x)}{\mathrm{d}\,x^2} = \frac{\mathrm{d}\,\phi'(x)}{\mathrm{d}\,x} \qquad \text{(definition)}$$

$$= \frac{\mathrm{d}\,(-x\,\phi(x))}{\mathrm{d}\,x} \qquad \text{(above result)}$$

$$= -\left[\phi(x) + x\,\phi'(x)\right] \qquad \text{(chain rule)}$$

$$= -\left[\phi(x) + x\left(-x\,\phi(x)\right)\right] \qquad \text{(above result)}$$

$$= \left[x^2 - 1\right]\phi(x)$$

3 Truncated and censored distributions

Truncation:

A truncated distribution is a conditional distribution that results from restricting the domain of some other probability distribution. Let X be a continuous r.v. with cdf $F_X(\cdot)$ and pdf $f_X(\cdot)$ over support $\text{Supp}(X) = \mathbb{R}$. Consider the distribution of X after restricting the support to some interval $(a, b]$, i.e., the distribution of $X \mid a < X \leq b$. Naturally, this distribution should have the same shape as the unrestricted distribution of X over $(a, b]$, but at the same time it must integrate to 1 over its support. This suggests dividing the density of X by the probability mass that $X \in (a, b]$. That is,

$$f_X(x \mid a < X \leq b) = \begin{cases} 0 & \text{if } \; x \leq a \\ \frac{f_X(x)}{\mathbb{P}(a < X \leq b)} & \text{if } \; a < x \leq b \\ 0 & \text{if } \; X > b \end{cases}$$

$$= \frac{\mathbb{I}\left[a < x \leq b\right] f_X(x)}{\mathbb{P}\left(a < X \leq b\right)}$$

$$= \frac{\mathbb{I}\left[a < x \leq b\right] f_X(x)}{F_X(b) - F_X(a)}$$

and we can verify that

$$\int_{-\infty}^{\infty} f_X(x \mid a < X \leq b)\, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathbb{I}\left[a < x \leq b\right] f_X(x)}{F_X(b) - F_X(a)}\, \mathrm{d}x$$

$$= \frac{1}{F_X(b) - F_X(a)} \int_a^b f_X(x)\, \mathrm{d}x$$

$$= \frac{1}{F_X(b) - F_X(a)} \left[F_X(b) - F_X(a)\right]$$

$$= 1$$

We can think of the special case of truncation from below —also known as truncation from the left— as $\Big\{X \mid X > a\Big\} = \lim\limits_{b \to \infty} \Big\{X \mid a < X \leq b\Big\}$ or, more precisely,

$$f_X(x \mid X > a) = \lim_{b \to \infty} \frac{\mathbb{I}\left[a < x \leq b\right] f_X(x)}{F_X(b) - F_X(a)}$$

$$= \frac{\mathbb{I}\left[x > a\right] f_X(x)}{1 - F_X(a)}$$

Obtaining the moments of a truncated distribution is standard once we know the relevant conditional pdf $f_X(x \mid a < X \leq b)$. For example, the mean is simply

$$\mathbb{E}\left[X \mid a < X \leq b\right] = \int_a^b x\, f_X(x \mid a < X \leq b)\, \mathrm{d}x$$

Analogous results can be obtained for

$$f_X(x \mid X < b) = \lim_{a \to -\infty} \frac{\mathbb{I}\left[a < x < b\right] f_X(x)}{F_X(b) - F_X(a)}$$

$$= \frac{\mathbb{I}\left[x < b\right] f_X(x)}{F_X(b)}$$

Censoring:

Censoring is a data problem or condition whereby values of the underlying random variable within an interval are observed, reported, or transformed into a specific value. For example, suppose that $Y^* \in \mathbb{R}_+$ is household income, but for some reason (maybe privacy concerns) high incomes are top-coded. That is, incomes above some threshold $Y^* \geq c$ are reported/observed as $Y = c$, while incomes below the threshold are completely observed. If income is above c, we know it is but not by how much. This is known as censoring from above or right censoring.

Now, thinking of the distribution that applies to Y, it seems evident that it is neither continuous nor discrete. In fact, it is a mixture of a continuous and a discrete distribution. Since

$$Y = \begin{cases} Y^* & \text{if } \; Y^* < c \\ c & \text{if } Y^* \geq c \end{cases},$$

the density for observations below c coincides with that of Y^*, while there is a mass point at $Y = c$ corresponding to $Y^* \geq c$ with probability mass $\mathbb{P}\left(Y^* \geq c\right) = 1 - F_{Y^*}(c)$, i.e,

$$f_Y(y) = \begin{cases} f_{Y^*}(y) & \text{if } Y < c \\ 1 - F_{Y^*}(c) & \text{if } Y = c \end{cases}$$

$$= f_{Y^*}(y)^{\mathbb{I}[Y<c]} \left[1 - F_{Y^*}(c)\right]^{1 - \mathbb{I}[y<c]}$$

$$\equiv f_{Y^*}(y)^{\delta} \left[1 - F_{Y^*}(c)\right]^{1-\delta}$$

It is straightforward to derive the moments once we know the censored pdf. For example, for the mean,

$$\begin{aligned}
\mathbb{E}_Y\left[Y\right] &= \mathbb{E}_\delta\Big[\mathbb{E}_{Y|\delta}\left[Y \mid \delta\right]\Big] && \text{(LIE)} \\
&= \mathbb{P}\left(\delta = 1\right)\mathbb{E}\left[Y \mid \delta = 1\right] + \mathbb{P}\left(\delta = 0\right)\mathbb{E}\left[Y \mid \delta = 0\right] && \text{(definition of expectation)} \\
&= \mathbb{P}\left(Y < c\right)\,\mathbb{E}\left[Y \mid Y < c\right] + \Big(1 - \mathbb{P}\left(Y < c\right)\Big)\,\mathbb{E}\left[Y \mid Y = c\right] && \text{(definition of } \delta\text{)} \\
&= F_{Y^*}(c)\int_{-\infty}^{c} y\, f_{Y^*}(y \mid Y^* < c)\,\mathrm{d}y + \Big(1 - F_{Y^*}(c)\Big)\,c && \text{(definition of } Y\text{)} \\
&= F_{Y^*}(c)\,\frac{\int_{-\infty}^{c} y\, f_{Y^*}(y)\,\mathrm{d}y}{F_{Y^*}(c)} + \Big(1 - F_{Y^*}(c)\Big)\,c && \text{(truncated distribution)} \\
&= \int_{-\infty}^{c} y\, f_{Y^*}(y)\,\mathrm{d}y + \Big(1 - F_{Y^*}(c)\Big)\,c
\end{aligned}$$

Similar arguments can be used to obtain analogous results for left censoring (or censoring from below), where

$$Y = \begin{cases} c & \text{if } Y^* \le c \\ Y^* & \text{if } Y^* > c \end{cases}$$

and

$$f_Y(y) = \begin{cases} F_{Y^*}(c) & \text{if } Y = c \\ f_{Y^*}(y) & \text{if } Y > c \end{cases}$$

4 Moments of the truncated normal distribution

Let $Y \sim \mathrm{N}\left(\mu, \sigma^2\right)$. Then, from our results in section 2,

$$f_Y(y) = \frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)$$

$$F_Y(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

We can use our general results for truncated distributions from section 3 to derive the pdf of $Y \mid Y > c$ for any constant $c \in \mathbb{R}$:

$$f_Y(y \mid Y > c) = \frac{\mathbb{I}\left[y > c\right] f_Y(y)}{1 - F_Y(c)}$$

$$= \begin{cases} 0 & \text{if } y \leq c \\ \frac{1}{\sigma}\frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} & \text{if } y > c \end{cases}$$

Using this result, the truncated mean is given by

$$\mathbb{E}\left[Y \mid Y > c\right] \equiv \int_{-\infty}^{\infty} y\, f_Y(y \mid Y > c)\, \mathrm{d}y \qquad \text{(definition)}$$

$$= \int_{c}^{\infty} y\, \frac{1}{\sigma}\frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)}\, \mathrm{d}y \qquad \text{(truncated density)}$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \int_{\frac{c-\mu}{\sigma}}^{\infty} \left(\mu + \sigma\, z\right)\, \frac{1}{\sigma}\phi\left(z\right)\sigma\, \mathrm{d}z \qquad \left(\text{c.o.v. } z = \frac{y-\mu}{\sigma}\right)$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\mu \int_{\frac{c-\mu}{\sigma}}^{\infty} \phi(z)\, \mathrm{d}z + \sigma \int_{\frac{c-\mu}{\sigma}}^{\infty} z\, \phi(z)\, \mathrm{d}z\right]$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\mu\left[1 - \Phi\left(\frac{c-\mu}{\sigma}\right)\right] + \sigma \int_{\frac{c-\mu}{\sigma}}^{\infty} -\mathrm{d}\phi(z)\right] \qquad \left(\phi'(z) = -z\phi(z)\right)$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\mu\left[1 - \Phi\left(\frac{c-\mu}{\sigma}\right)\right] - \sigma\,\phi(z)\Big|_{\frac{c-\mu}{\sigma}}^{\infty}\right]$$

. . . continues . . .

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\mu\left[1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right] - \sigma\left[0-\phi\left(\frac{c-\mu}{\sigma}\right)\right]\right]$$

$$= \mu + \sigma \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)}$$

Similarly,

$$\mathbb{E}\left[Y^2 \mid Y > c\right] \equiv \int_{-\infty}^{\infty} y^2\, f_Y(y \mid Y > c)\, \mathrm{d}y \qquad \text{(definition)}$$

$$= \int_{c}^{\infty} y^2\, \frac{1}{\sigma} \frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)}\, \mathrm{d}y \qquad \text{(truncated density)}$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \int_{\frac{c-\mu}{\sigma}}^{\infty} (\mu + \sigma\, z)^2\, \frac{1}{\sigma}\phi(z)\, \sigma\, \mathrm{d}z \qquad \left(\text{c.o.v. } z = \frac{y-\mu}{\sigma}\right)$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\mu^2 \int_{\frac{c-\mu}{\sigma}}^{\infty} \phi(z)\, \mathrm{d}z + 2\,\mu\,\sigma \int_{\frac{c-\mu}{\sigma}}^{\infty} z\, \phi(z)\, \mathrm{d}z + \sigma^2 \int_{\frac{c-\mu}{\sigma}}^{\infty} z^2\, \phi(z)\, \mathrm{d}z\right]$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\left(\mu^2+\sigma^2\right) \int_{\frac{c-\mu}{\sigma}}^{\infty} \phi(z)\, \mathrm{d}z + 2\,\mu\,\sigma \int_{\frac{c-\mu}{\sigma}}^{\infty} z\, \phi(z)\, \mathrm{d}z + \sigma^2 \int_{\frac{c-\mu}{\sigma}}^{\infty} \left(z^2-1\right)\, \phi(z)\, \mathrm{d}z\right]$$

$$= \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[\left(\mu^2+\sigma^2\right)\left[1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right] - 2\,\mu\,\sigma\,\phi\left(\frac{c-\mu}{\sigma}\right) + \sigma^2 \int_{\frac{c-\mu}{\sigma}}^{\infty} \mathrm{d}\phi'(z)\right] \qquad \left(\phi''(z) = \left[z^2-1\right]\phi(z)\right)$$

. . . continues . . .

$$= \mu^2 + \sigma^2 + \sigma \, \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[2\,\mu\,\phi\left(\frac{c-\mu}{\sigma}\right) + \sigma\,\phi'(z) \Big|_{\frac{c-\mu}{\sigma}}^{\infty} \right]$$

$$= \mu^2 + \sigma^2 + \sigma \, \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \left[2\,\mu\,\phi\left(\frac{c-\mu}{\sigma}\right) + \sigma \Big(-z\phi(z) \Big) \Big|_{\frac{c-\mu}{\sigma}}^{\infty} \right] \qquad (\phi'(z) = -z\phi(z))$$

$$= \mu^2 + \sigma^2 + \sigma \, \frac{1}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \Bigg[2\,\mu\,\phi\left(\frac{c-\mu}{\sigma}\right) - \sigma \left[0 - \frac{c-\mu}{\sigma}\phi\left(\frac{c-\mu}{\sigma}\right) \right] \Bigg]$$

$$= \mu^2 + \sigma^2 + \sigma \, (c+\mu) \, \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)}$$

Therefore,

$$\mathrm{Var}\left(Y \mid Y > c\right) = \mathbb{E}\left[Y^2 \mid Y > c\right] - \mathbb{E}\left[Y \mid Y > c\right]^2 \qquad \text{(variance property)}$$

$$= \mu^2 + \sigma^2 + \sigma \, (c+\mu) \, \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} - \left(\mu + \sigma \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \right)^2 \qquad \text{(above results)}$$

$$= \sigma\, c \, \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} + \sigma^2 \left[1 - \left(\frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \right)^2 \right]$$

$$= \sigma^2 \left[1 + \frac{c}{\sigma} \frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \right)^2 \right]$$