Simultaneous Equations Models*

EC309 – Lent term

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Contents

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1 Notation

Consider the following system of N equations in N endogenous variables and K exogenous variables

$$\begin{aligned} a_1^1\, y_{1t} + \cdots + a_N^1\, y_{Nt} + b_1^1\, x_{1t} + \cdots + b_K^1\, x_{Kt} &= u_{1t} \\ &\vdots \\ a_1^N\, y_{1t} + \cdots + a_N^N\, y_{Nt} + b_1^N\, x_{1t} + \cdots + b_K^N\, x_{Kt} &= u_{Nt}, \end{aligned} \tag{1}$$

and a sample of T *iid* observations $t \in \{1, \ldots, T\}$.

Let

$$y_t \equiv \begin{pmatrix} y_{1t} \\ \vdots \\ y_{Nt} \end{pmatrix}_{N\times 1} \quad x_t \equiv \begin{pmatrix} x_{1t} \\ \vdots \\ x_{Kt} \end{pmatrix}_{K\times 1} \quad u_t \equiv \begin{pmatrix} u_{1t} \\ \vdots \\ u_{Nt} \end{pmatrix}_{N\times 1}$$

$$A \equiv \begin{bmatrix} a_1^1 & \cdots & a_N^1 \\ \vdots & \ddots & \vdots \\ a_1^N & \cdots & a_N^N \end{bmatrix}_{N\times N} \equiv \begin{bmatrix} {a^1}'_{1\times N} \\ \vdots \\ {a^N}'_{1\times N} \end{bmatrix}_{N\times N}$$

$$B \equiv \begin{bmatrix} b_1^1 & \cdots & b_K^1 \\ \vdots & \ddots & \vdots \\ b_1^N & \cdots & b_K^N \end{bmatrix}_{N\times K} \equiv \begin{bmatrix} {b^1}'_{1\times K} \\ \vdots \\ {b^N}'_{1\times K} \end{bmatrix}_{N\times K}.$$

Then, we can compactly write the system for observation t as

$$Ay_t + Bx_t = u_t \tag{2}$$

and, transposing and stacking the system for the T observations, we obtain the matrix equation

$$YA' + XB' = U, \tag{3}$$

where

$$Y \equiv \begin{bmatrix} y_{11} & \cdots & y_{N1} \\ \vdots & \ddots & \vdots \\ y_{1T} & \cdots & y_{NT} \end{bmatrix}_{T\times N} \equiv \begin{bmatrix} y'_{1\,1\times N} \\ \vdots \\ y'_{T\,1\times N} \end{bmatrix}_{T\times N}$$

$$X \equiv \begin{bmatrix} x_{11} & \cdots & x_{K1} \\ \vdots & \ddots & \vdots \\ x_{1T} & \cdots & x_{KT} \end{bmatrix}_{T\times K} \equiv \begin{bmatrix} x'_{1\,1\times K} \\ \vdots \\ x'_{T\,1\times K} \end{bmatrix}_{T\times K}$$

$$U \equiv \begin{bmatrix} u_{11} & \cdots & u_{N1} \\ \vdots & \ddots & \vdots \\ u_{1T} & \cdots & u_{NT} \end{bmatrix}_{T\times N} \equiv \begin{bmatrix} u'_{1\,1\times N} \\ \vdots \\ u'_{T\,1\times N} \end{bmatrix}_{T\times N} .$$

2 Identification

Solving Equation (2) for y_t, we get the reduced form

$$y_t = \Pi x_t + v_t, \tag{4}$$

where

$$\Pi \equiv -A^{-1}B$$

$$v_t \equiv A^{-1}u_t.$$

Since x_t are exogenous variables —i.e., $\mathbb{E}\left[x_t u_t'\right] = 0_{K\times N}$—, Π is identified if $\operatorname{rank}\left(\mathbb{E}\left[x_t x_t'\right]\right) = K$:

$$\begin{aligned} & y_t' = x_t'\,\Pi' + v_t' \\ \implies \quad & x_t\,y_t' = x_t\,x_t'\,\Pi' + x_t\,v_t' \\ \implies \quad & \mathbb{E}\left[x_t\,y_t'\right] = \mathbb{E}\left[x_t\,x_t'\right]\,\Pi' + \mathbb{E}\left[x_t\,u_t'\right]\,A^{-1\prime} \\ \iff \quad & \Pi' = \mathbb{E}\left[x_t\,x_t'\right]^{-1}\,\mathbb{E}\left[x_t\,y_t'\right] \end{aligned}$$

Therefore, identification of the structural parameters in A and B boils down to whether matrix equation

$$\Pi = -A^{-1}\,B \iff A\,\Pi + B = 0_{N\times K} \tag{5}$$

has a unique solution. It is immediately obvious that this is not the case without further restrictions on the structural parameters —i.e., additional linear equations in the structural parameters— since there are N^2 parameters in A and NK parameters in B but we have only NK linear equations in (5).

We can understand the problem better by vectorizing Equation (5). Write

$$\Pi = \begin{bmatrix} \pi_{11} & \cdots & \pi_{K1} \\ \vdots & \ddots & \vdots \\ \pi_{1N} & \cdots & \pi_{KN} \end{bmatrix}_{N \times K}$$

and define

$$a \equiv \operatorname{vec}(A) = \begin{pmatrix} \begin{pmatrix} a_1^1 \\ \vdots \\ a_1^N \end{pmatrix} \\ \vdots \\ \begin{pmatrix} a_N^1 \\ \vdots \\ a_N^N \end{pmatrix} \end{pmatrix}_{N^2 \times 1}$$

$$b \equiv \operatorname{vec}(B) = \begin{pmatrix} \begin{pmatrix} b_1^1 \\ \vdots \\ b_1^N \end{pmatrix} \\ \vdots \\ \begin{pmatrix} b_K^1 \\ \vdots \\ b_K^N \end{pmatrix} \end{pmatrix}_{NK \times 1}$$

$$\alpha \equiv \operatorname{vec}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = \begin{bmatrix} \operatorname{vec}(A) \\ \operatorname{vec}(B) \end{bmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}_{(N^2+NK)\times 1}$$

$$V \equiv \begin{bmatrix} \Pi' \otimes I_N & I_{NK} \end{bmatrix}$$

$$= \left[\left(\begin{bmatrix} \pi_{11} & \cdots & \pi_{1N} \\ \vdots & \ddots & \vdots \\ \pi_{K1} & \cdots & \pi_{KN} \end{bmatrix}_{K \times N} \otimes \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{N \times N} \right) \quad \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{NK \times NK} \right]$$

$$
= \begin{bmatrix}
\begin{bmatrix} \pi_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{11} \end{bmatrix} & \cdots & \begin{bmatrix} \pi_{1N} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{1N} \end{bmatrix} & I_N & \cdots & 0_{N\times N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\begin{bmatrix} \pi_{K1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{K1} \end{bmatrix} & \cdots & \begin{bmatrix} \pi_{KN} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{KN} \end{bmatrix} & 0_{N\times N} & \cdots & I_N
\end{bmatrix}_{NK\times(N^2+NK)} .
$$

Now, vectorize the LHS of Equation (5):[1]

$$
\begin{aligned}
\text{vec}\,(A\,\Pi + B) &= \text{vec}\,(A\,\Pi) + \text{vec}\,(B) \\
&= \text{vec}\,(I_N A\,\Pi) + I_{NK}\,\text{vec}\,(B) \\
&= \big(\Pi' \otimes I_N\big)\,\text{vec}\,(A) + I_{NK}\,\text{vec}\,(B) \\
&= \big(\Pi' \otimes I_N\big)\;a + I_{NK} b \\
&= \Big[(\Pi' \otimes I_N) \quad I_{NK}\Big] \begin{pmatrix} a \\ b \end{pmatrix} \\
&= V\alpha.
\end{aligned}
$$

Therefore, Equation (5) can be written as the system of NK linear equations in $N^2 + NK$ unknowns —the parameters in α—

$$
V\alpha = 0_{NK\times 1} \tag{6}
$$

or, equivalently,

$$
\begin{aligned}
\pi_{11}\, a_1^1 + \cdots + \pi_{1N}\, a_N^1 + b_1^1 &= 0 \\
&\vdots \\
\pi_{11}\, a_1^N + \cdots + \pi_{1N}\, a_N^N + b_1^N &= 0 \\
&\vdots
\end{aligned} \tag{7}
$$

[1] The first equality uses the linearity of the vec operator, and the third equality uses the property that, for arbitrary conformable matrices A, B, and C,

$$
\text{vec}\,(ABC) = \big(C' \otimes A\big)\,\text{vec}\,(B).
$$

$$\pi_{K1}\, a_1^1 + \cdots + \pi_{KN}\, a_N^1 + b_K^1 = 0$$

$$\vdots$$

$$\pi_{K1}\, a_1^N + \cdots + \pi_{KN}\, a_N^N + b_K^N = 0.$$

It is clear that we need N^2 additional linearly independent restrictions on the parameters in α to have identification —i.e., a system with a unique solution.

Consider imposing the set of L linear restrictions on α

$$W\alpha = w,$$

where W is an $L \times (N^2 + NK)$ matrix and w is an $L \times 1$ vector. Then our system becomes

$$\begin{bmatrix} V \\ W \end{bmatrix} \alpha = \begin{pmatrix} 0_{NK\times 1} \\ w \end{pmatrix}, \tag{8}$$

and a *necessary* condition for a unique solution, which we call the **order condition**, is that we have at least as many linearly independent equations as unknowns, i.e.,

$$NK + L \geq N^2 + NK \iff L \geq N^2.$$

In other words, matrix W must have at least N^2 rows. A *sufficient* condition for a unique solution, which we call the **rank condition** is that

$$\text{rank}\left(\begin{bmatrix} V \\ W \end{bmatrix}\right) = N^2 + NK.$$

An alternative way to understand the problem of identification is the following. Let P be any $N \times N$ nonsingular matrix and premultiply Equation (2) by P:

$$A^* y_t + B^* x_t = u_t^*, \tag{9}$$

where $A^* \equiv PA$, $B^* \equiv PB$, and $u_t^* \equiv Pu_t$. Notice that the reduced form associated with (9),

$$\begin{aligned} y_t &= -A^{*-1} B^* x_t + A^{*-1} u_t^* \\ &= -(PA)^{-1}(PB)x_t + (PA)^{-1}(Pu_t) \\ &= -A^{-1}P^{-1}PBx_t + A^{-1}P^{-1}Pu_t \\ &= -A^{-1}Bx_t + A^{-1}u_t \end{aligned}$$

$$= \Pi x_t + v_t,$$

is exactly (4). Therefore, there are multiple equivalent structures —i.e., structural systems of equations like (1) and (9)— in the sense that they share the same reduced form. That is, one reduced form is consistent with multiple structures —actually, infinitely many: one for each nonsingular P.

We can define an **admissible linear transformation** as an $N \times N$ non singular matrix such that $P \begin{bmatrix} A & B \end{bmatrix}$ satisfies all the restrictions we imposed on $\begin{bmatrix} A & B \end{bmatrix}$.[2] Identification is equivalent to I_N being the only admissible linear transformation so that there is a unique structure that is consistent with a given reduced form. We need "prior information" on the structural parameters —i.e., additional restrictions— in order to identify the system. This is easier to understand in the special case where we are interested in the structural parameters from the first equation only.

2.1 Identification of the parameters in a single equation

Suppose we are only interested in the first equation of system (1)

$$a_1^1 \, y_{1t} + \cdots + a_N^1 \, y_{Nt} + b_1^1 \, x_{1t} + \cdots + b_K^1 \, x_{Kt} = u_{1t}$$

$$\iff \; y_t' a^1 + x_t' b^1 = u_{1t}.$$

Then, the question is when is

$$\alpha^1 \equiv \begin{pmatrix} a^1 \\ b^1 \end{pmatrix} = \begin{pmatrix} a_1^1 \\ \vdots \\ a_N^1 \\ b_1^1 \\ \vdots \\ b_K^1 \end{pmatrix}$$

identified, and the answer boils down to whether the system of equations corresponding to the first row of Equation (5)

$$a^{1\prime} \, \Pi + b^{1\prime} = 0_{1 \times K}$$

$$\iff \qquad \Pi' \, a^1 + b^1 = 0_{K \times 1}$$

[2] We would also require that $\mathbb{E}\left[u_t^* u_t^{*\prime}\right] = P \Sigma P'$ satisfies all the restrictions on $\Sigma \equiv \mathbb{E}\left[u_t u_t'\right]$ but, for simplicity, we are abstracting away from such restrictions.

$$\iff \begin{bmatrix}\Pi' & I_K\end{bmatrix}\begin{pmatrix}a^1\\ b^1\end{pmatrix} = 0_{K\times 1}$$

$$\iff \qquad V^1\alpha^1 = 0_{K\times 1}$$

has a unique solution. This is a system of K linear equations on $N+K$ unknowns —the parameters in α^1—, where

$$V^1 \equiv \begin{bmatrix}\Pi' & I_K\end{bmatrix}.$$

We need at least N additional linearly independent equations —i.e., restrictions on the parameters in α^1— to be able to identify α^1. If we consider imposing the set of L_1 linearly independent restrictions

$$W^1\alpha^1 = w^1 \tag{10}$$

where W^1 is $L_1 \times (N+K)$ and w^1 is $L_1 \times 1$, the *necessary* **order condition** is that

$$K + L_1 \geq N + K \iff L_1 \geq N$$

The system becomes

$$\begin{bmatrix}V^1\\ W^1\end{bmatrix}\alpha^1 = \begin{pmatrix}0_{K\times 1}\\ w^1\end{pmatrix}.$$

Let

$$C \equiv \begin{bmatrix}A & B\end{bmatrix} \iff C' \equiv \begin{bmatrix}A'\\ B'\end{bmatrix}$$

be the matrix of structural parameters in system (1) and $\{e_i\}_{i=1}^N$ the canonical basis vectors of $\mathbb{R}^N$. Note that the i^{th} column of C' corresponds to the structural parameters that appear in the i^{th} equation of system (1), α^i:

$$\alpha^1 = \begin{bmatrix}\begin{bmatrix}a_1^1 & \cdots & a_1^N\\ \vdots & \ddots & \vdots\\ a_N^1 & \cdots & a_N^N\end{bmatrix}\\ \begin{bmatrix}b_1^1 & \cdots & b_1^N\\ \vdots & \ddots & \vdots\\ b_K^1 & \cdots & b_K^N\end{bmatrix}\end{bmatrix}\begin{pmatrix}1\\ \vdots\\ 0\end{pmatrix}_{N\times 1}$$

$$= \begin{bmatrix} A' \\ B' \end{bmatrix} e_1$$

$$= C'e_1$$

and, in general, $\alpha^i = C'e_i$. Thus,

$$C' = C'I_N$$
$$= C' \begin{bmatrix} e_1 & \cdots & e_N \end{bmatrix}$$
$$= \begin{bmatrix} C'e_1 & \cdots & C'e_N \end{bmatrix}$$
$$= \begin{bmatrix} \alpha^1 & \cdots & \alpha^N \end{bmatrix}.$$

We define an **admissible linear transformation** as an $N \times N$ non singular matrix P such that $C'P$ satisfies all the restrictions we imposed on α —the parameters in C— through the set of restrictions in (10). Partition arbitrary nonsingular $N \times N$ matrix P as

$$P = \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}$$

so that

$$\widetilde{C}' \equiv C'P$$
$$= C' \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}$$
$$= \begin{bmatrix} C'p_1 & \cdots & C'p_N \end{bmatrix}$$
$$\equiv \begin{bmatrix} \widetilde{\alpha}^1 & \cdots & \widetilde{\alpha}^N \end{bmatrix}.$$

The additional information about the structural parameters contained in the linear restrictions (10) is sufficient for identification of the structural parameters in the first equation, α^1, if the only $\widetilde{\alpha}^1$ satisfying (10) is $\widetilde{\alpha}^1 = \alpha^1$, i.e., the only admissible linear transformations P are such that $p_1 = e_1$.

Now, $\widetilde{\alpha}^1$ satisfies the linear restrictions in (10) if and only if

$$W^1 \widetilde{\alpha}^1 = w^1$$

$$\iff W^1 \left(C' p_1\right) = w^1$$

$$\iff \left(W^1 C'\right) p_1 = w^1,$$

which is true by assumption for $p_1 = e_1$ since

$$W^1 \left(C' e_1\right) = W^1 \alpha^1$$

$$= w^1$$

by (10). Therefore, the solution set of the $L_1 \times N$ linear system —with $L_1 \geq N$—

$$\left(W^1 C'\right) p_1 = w^1$$

is nonempty and what we are looking for is a condition that guarantees its dimension is 1. We know from linear algebra that the system will have at most one solution if

$$\operatorname{rank}\left(W^1 C'\right) = N$$

and, since we know it has at least one solution, this **rank condition** is sufficient for identification.